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On Rank-Two and Affine Cluster Algebras

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Abstract

Motivated by existing results about the Kronecker cluster algebra, this thesis is concerned with two families of cluster algebras, which are two different ways of generalizing the Kronecker case: rank-two cluster algebras, and cluster algebras of type $\widetilde{A}_{n,1}$. Regarding rank-two cluster algebras, our main result is a conjectural bijection that would prove the equivalence of two combinatorial formulas for cluster variables of rank-two skew-symmetric cluster algebras. We identify a technical result that implies the bijection and make partial progress towards its proof. We then shift gears to study certain power series which arise as limits of ratios of F-polynomials in cluster algebras of type $\widetilde{A}_{n,1}$. With several different perspectives in mind, including that of continued fractions, path-ordered products and the surface model, we state and prove various equivalent formulas for these power series. In our study of these two families, we make use of a product formula for F-polynomials, called Gupta's formula, which is applicable to all cluster algebras of geometric type. We dedicate one of our chapters to an exposition of this formula. Though Gupta's formula has previously appeared in different notations, and in that sense is not new, we believe that our statement and proof of the formula provides a new approach to the formula which is elementary and combinatorial.

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Acknowledgments

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Chapter 1

Introduction

Cluster algebras are certain commutative algebras with a rich combinatorial structure, first introduced in Fomin and Zelevinsky (2002). Their inception was motivated by the study of semicanonical bases of Lie algebras; but since then, researchers have made deep connections between cluster algebras and many other areas of math and physics, including discrete dynamical systems, Poisson geometry, higher Teichmüller spaces, commutative and non-commutative algebraic geometry, string theory, and quiver representation theory; see Keller (2012) for specific references. The interdisciplinary nature of the cluster algebras community and the interconnectedness of different perspectives on cluster algebras is one of the reasons that research in this area can feel incredibly exciting. A quick introduction to cluster algebras is presented in Chapter 2.

The starting point of this thesis is the cluster algebra defined by the Kronecker quiver (Figure 1.1). We will refer to this cluster algebra as the Kronecker cluster algebra.

$$1 \rightrightarrows 2$$

Figure 1.1 The Kronecker Quiver K_2

The Kronecker cluster algebra is arguably the simplest cluster algebra of *infinite type* (it has infinitely many distinguished generators, or in cluster-algebra-speak, infinitely many *cluster variables*). Perhaps consequently, it is very well-studied. To mention a few references, explicit formulas for a family of associated Laurent polynomials, called the semicanonical bases, are given in Caldero and Zelevinsky (2006). This in particular includes explicit formulas for the cluster variables, which can be translated to explicit formulas for certain polynomials (called *F-polynomials*) associated to the cluster variables. Musiker and Propp (2007) gives a combinatorial interpretation of the coefficients of the cluster variables in terms of enumeration of the perfect matchings of certain graphs, which makes the positivity of coefficients apparent. As an example of his combinatorial approach to scattering diagrams, Reading (2020b) fully studies the scattering diagram for the Kronecker cluster algebra and computes two distinguished infinite path-ordered products.

With the Kronecker quiver as the starting point, our thesis has gone in two directions. One is the study of rank-two cluster algebras. These cluster algebras are defined by two parameters $b, c > 0$, so we refer to the cluster algebra defined by b and c as $\mathcal{A}(b, c)$. When $b = c = r$, they are equivalently defined by r-Kronecker quiver K_r, pictured in Figure 1.2. In particular, the Kronecker cluster algebra is $\mathcal{A}(2, 2)$, defined by the Kronecker quiver K_2.

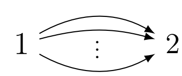


Figure 1.2 The r-Kronecker Quiver K_r, which has r edges from 1 to 2

There is a large amount of literature on rank-two cluster algebras (see Lee (2012), Lee and Schiffler (2013), Lee et al. (2014), Gupta (2018), and Cheung et al. (2017)). Compared to the Kronecker case where cluster variables have simple formulas, when $bc > 4$, we only have formulas for cluster variables in terms of combinatorial objects which are themselves not fully understood. Two natural questions that arise are:

1. Can we come up with simpler formulas for the cluster variables or, equivalently, F-polynomials of $\mathcal{A}(b, c)$ when $bc > 4$?

2. How do existing formulas compare to each other?

Gupta's formula, detailed in Chapter 3, provides a formula for the coefficients of F-polynomials as a sum over a (possibly infinite) set of tuples. During the UMN REU in the summer of 2020, this seemed like a possible point of entry to some progress on the first question. In my REU report, I show the agreement between Gupta's formula for F-polynomials of the Kronecker cluster algebra and the well-known formula proven in Caldero and Zelevinsky (2006), but it was difficult to generalize this to the r-Kronecker case for $r > 2$.

As we turned to Lee and Schiffler (2013) and Lee et al. (2014) for inspiration, we made progress on Question 2 independently of Gupta's formula. Chapter 4 first gives an exposition on rank-two cluster algebras, specializing the general definitions given in Chapter 2 to the rank-two case. We then briefly introduce the results of Lee and Schiffler (2013) and Lee et al. (2014): the main result of Lee and Schiffler (2013) is a combinatorial formula for cluster variables in $\mathcal{A}(r,r)$, and Lee et al. (2014) gives a formula for certain *greedy elements* in $\mathcal{A}(b,c)$, which include cluster variables. As an exercise on the understanding of Lee and Schiffler (2013), we show that Lee and Schiffler's formula reduces to the simple formula of Caldero and Zelevinsky (2006) in the Kronecker case. To reach their main result, Lee et al. (2014) proves that a certain polygon contains the support of *greedy elements* (cluster variables are in particular greedy elements). We check that when we specialize their proposition to cluster variables of the r-Kronecker, their support polygon agrees with my conjecture presented in Lin, Feiyang (2020). Lastly, assuming a conjectural identity related to maximal Dyck paths, we prove a weight-preserving bijection between the combinatorial objects of Lee and Schiffler (2013) and Lee et al. (2014).

In the other direction, we study a different generalization of the Kronecker case, namely the cluster algebra $\widetilde{A}_{n,1}$, which is defined by the following quiver.

$$\begin{tikzcd} 1 \arrow[r] \arrow[rrr, bend left] & 2 \arrow[r] & \cdots \arrow[r] & n+1 \end{tikzcd}$$

Figure 1.3 The $Q_{n,1}$ Quiver

Notice that the Kronecker cluster algebra can be written as $\widetilde{A}_{1,1}$. The cluster algebra $\widetilde{A}_{n,1}$ is better-behaved than $\mathcal{A}(b,c)$ when $bc > 4$ because they correspond to affine root systems rather than root systems of indefinite type (we touch on this in Section 2.4 of Chapter 2). In this thesis, we focus on certain power series that arise as limits of ratios of F-polynomials in $\widetilde{A}_{n,1}$. The same limit when $n = 1$ is studied in both Reading (2020b) and Canakci and Schiffler (2017). In Chapter 5, we take a variety of approaches to study these limits, such as via their coefficients, continued fraction expansions, generating functions, and product forms given by Gupta's formula. For cluster algebras of infinite type, one is tempted to look for ways to capture the limit behavior of the cluster algebra. The limit of ratios of F-polynomials is one way of doing so. As such, we hope that our thorough study of the case of $\widetilde{A}_{n,1}$ provides an example for the research community beyond the Kronecker cluster algebra.

Another contribution of this thesis is an exposition of Gupta's formula for F-polynomials in skew-symmetrizable cluster algebras, which first appeared in Gupta (2018) in a different form. Her formula in the skew-symmetric case was later rewritten into its current form and reproven by Gregg Musiker; this work was presented at an AMS sectional meeting, but has not appeared publicly in the literature. We generalize Musiker's work slightly to the skew-symmetrizable case and discuss the connections between Gupta's formula and previously existing work. We hope our account will make Gupta's formula more accessible to the cluster algebra community.

Chapter 2

Background on Cluster Algebras

The first two sections of this chapter define cluster algebras and various parameterizations of cluster variables. The next section introduces root systems and some results relating root systems to cluster algebras, with more attention to the finite and affine cases. We then define scattering diagrams and path-ordered products, which is another piece of evidence that it is fruitful to understand cluster algebras from a root-theoretic perspective.

2.1 What is a cluster algebra?

Algebraic structures that we are most familiar with, such as groups, ideals, vector spaces, are most commonly defined by specifying a complete list of generators and the relations among them. Cluster algebras are quite different in this sense. Instead of specifying all the generators, one defines a cluster algebra by starting with a set of generators $x_1, \ldots, x_n$ and a rule for making other generators from existing ones. A cluster algebra is then defined to be the commutative subalgebra of the ring of rational functions in $x_1, \ldots, x_n$ generated by the generators. The generators of the cluster algebra thus produced are called *cluster variables* and they are grouped into *clusters*. We will elaborate on the precise definition in the remainder of this section.

There are a few levels of generality, but for this thesis we will restrict to the case of cluster algebras of geometric type, which means that the iterative procedure we use to produce cluster variables can be encoded by a matrix. We also restrict to the case of *principal coefficients*, which is a canonical way of setting the initial conditions for this iterative procedure.

Let n be a positive integer. Let $\mathbb{P} = \mathrm{Trop}(y_1, \dots, y_n)$ be the semifield whose multiplicative group is the free abelian group generated by $y_1, \dots, y_n$, endowed with an auxillary addition $\oplus$ defined by

$$\prod_j y_j^{a_j} \oplus \prod_j y_j^{b_j} = \prod_j y_j^{\min(a_j, b_j)}, \text{ for } a_j, b_j \in \mathbb{Z}.$$

Thus we may understand the multiplicative part of $\mathbb{P}$ as the group of Laurent monomials in $y_1, \dots, y_n$ under multiplication.

For $\Bbbk$ a field, let $\Bbbk(x_1, \dots, x_n)$ be the field of Laurent polynomials in the variables $x_1, \dots, x_n$. A cluster algebra (of geometric type, with principal coefficients) is a subalgebra of an ambient field $\mathcal{F} = \mathbb{QP}(x_1, \dots, x_n) \cong \mathbb{Q}(x_1, \dots, x_n, y_1, \dots, y_n)$. To define its generators, we start with the data of a skew-symmetrizable matrix B_o called the *initial exchange matrix.*

Let $[n] = \{1, 2, \dots, n\}$.

Definition 2.1.1 (Skew-symmetric/symmetrizable)**.** An n-by-n matrix $B = (b_{ij})$ is *skew-symmetric* if $B^T = -B$, or equivalently, $b_{ij} = -b_{ji}$ for all $i, j \in [n]$. It is *skew-symmetrizable* if there exists a diagonal matrix D with positive entries such that DB is skew-symmetric; equivalently, B is skew-symmetrizable if there exists $\delta_i > 0$ for $i \in [n]$ such that $\delta_i b_{ij} = -\delta_j b_{ji}$. If B is skew-symmetric, letting D be the identity matrix establishes that B is skew-symmetrizable.

Definition 2.1.2 (Cluster Seed)**.** A *cluster seed* is a tuple $\Sigma_t = (\mathbf{x}, \mathbf{y}, B_t)$, where

- $B_t = (b_{ij}^t)$ is an n-by-n skew-symmetrizable matrix with integer entries;
- $\mathbf{x} = (x_{1;t}, x_{2;t}, \dots, x_{n;t})$ where each $x_{k;t} \in \mathcal{F}$ is called a *cluster variable*;
- $\mathbf{y} = (y_{1;t}, y_{2;t}, \dots, y_{n;t})$ where each $y_{k;t} \in \mathbb{P}$ is called a *coefficient variable.*

Let $\mathbb{T}_n$ be the n-regular tree where at each vertex, the n edges emanating from it are labeled by $1, \dots, n$. For reasons that will become clear in a moment, we parameterize cluster seeds by vertices $t \in \mathbb{T}_n$. The initial exchange matrix allows us to define an *initial cluster seed* Σ_{t_o}, on which we perform *mutations* to get other cluster seeds. The *initial cluster seed* is $\Sigma_{t_o} = (\mathbf{x}, \mathbf{y}, B_o)$, where for $1 \leq i \leq n$, $x_{i;t_o} = x_i$ and $y_{i;t_o} = y_i$.

Denote by $[a]_+ = \max(a, 0)$.

Definition 2.1.3 (Seed mutation). Let $t, t' \in \mathbb{T}_n$ be connected by an edge labeled k. We may mutate a cluster seed $\Sigma_t = (\mathbf{x}, \mathbf{y}, B_t)$ in direction $k \in [n]$ to obtain a different cluster seed $\Sigma_{t'} = (\mathbf{x}', \mathbf{y}', B_{t'})$. Let $B_t = (b_{ij})$. Then the components of the new seed are defined as follows:

- $B_{t'} = (b^{t'}_{ij}) = (b'_{ij})$, where

  $b'_{ij} = -b_{ij}$ if $i = k$ or $j = k$,

  $b'_{ij} = b_{ij} + b_{ik}b_{kj}$ if $b_{ik}, b_{kj} > 0$,

  $b'_{ij} = b_{ij} - b_{ik}b_{kj}$ if $b_{ik}, b_{kj} < 0$, and

  $b'_{ij} = b_{ij}$ otherwise;

- $\mathbf{x}' = (x_{1;t'}, x_{2;t'}, \ldots, x_{n;t'})$ where $x_{i;t'} = x_{i;t}$ if $i \neq k$ and

$$x_{k;t'} = \frac{y_{k;t} \prod_j x_{j;t}^{[b^t_{jk}]_+} + \prod_j x_{j;t}^{[-b^t_{jk}]_+}}{(y_{k;t} \oplus 1)x_{k;t}};$$

- $\mathbf{y}' = (y_{1;t'}, y_{2;t'}, \ldots, y_{n;t'})$, where $y_{k;t'} = y_{k;t}^{-1}$, and if $j \neq k$,

$$y_{j;t'} = y_{j;t} y_{k;t}^{[b^t_{kj}]_+} (y_{k;t} \oplus 1)^{-b^t_{kj}}.$$

We often denote mutation in direction k by μ_k, and write $\Sigma_{t'} = \mu_k \Sigma_t$. We use the convention of applying mutations left to right. For example, $\mu_1\mu_2\Sigma$ means apply μ_1 to Σ, and then apply μ_2.

With some algebra, we can check that $\mu_k^2 = 1$ for any $k \in [n]$. We still need to make sure that this association of a vertex t with a cluster seed Σ_t is well-defined. Notice that each vertex of $\mathbb{T}_n$ is connected to t_o by a unique simple path, which gives rise to a canonical sequence of mutations that we can apply to the initial seed to obtain a seed for each $t \in \mathbb{T}_n$. To check that any other way of obtaining Σ_t agrees with the current labeling of t, it suffices to check that seeds assigned to t and t' are related by μ_k if t and t' are connected by an edge labeled by k. This follows from the fact that the simple paths from t_o to t and t' must differ by an edge labeled k. Note that it is possible for two different vertices to be labeled with identical seeds. In general, in addition to $\mu_i^2 = 1$, there are other relations between the mutations.

We are now ready to define cluster algebras.

Definition 2.1.4. Let B_o be a skew-symmetrizable matrix which is n-by-n. Let $\mathcal{T}_n$ be the n-regular tree and associate seeds Σ_t to each vertex $t \in \mathcal{T}_n$ according to the mutation rules described above, starting at the initial seed defined by B_o. Then the *cluster algebra* with initial exchange matrix B_o is the subalgebra of $\mathcal{F} = \mathbb{QP}(x_1, \ldots, x_n)$ generated by the cluster variables:

$$\mathcal{A} = \mathbb{Z}[\{x_{i;t}, t \in \mathbb{T}_n\}],$$

where $x_{i;t}$ is the i-th cluster variable of the cluster Σ_t. We say that $\mathcal{A}$ has *rank n*.

We now try our hands on some examples.

Example 2.1.5 (Cluster Algebra of Type A_2)**.** Consider the cluster algebra $\mathcal{A}$ with initial exchange matrix

$$B_o = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Figure 2.1 shows the cluster seeds along the mutation sequence $\mu_1\mu_2\mu_1\mu_2\mu_1$. At each seed, we have $\mathbf{x}$ and then $\mathbf{y}$ below the exchange matrix.

$$\begin{tikzcd}
\begin{matrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ (x_1, x_2) \\ (y_1, y_2)\end{matrix} \arrow[r, "\mu_1"] & \begin{matrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ (\frac{y_1+x_2}{x_1}, x_2) \\ (y_1^{-1}, y_1y_2)\end{matrix} \arrow[r, "\mu_2"] & \begin{matrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ (\frac{y_1+x_2}{x_1}, \frac{y_1y_2x_1+y_1+x_2}{x_1x_2}) \\ (y_2, y_1^{-1}y_2^{-1})\end{matrix} \arrow[d, "\mu_1"] \\
\begin{matrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ (x_2, x_1) \\ (y_2, y_1)\end{matrix} & \begin{matrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ (\frac{1+y_2x_1}{x_2}, x_1) \\ (y_2^{-1}, y_1)\end{matrix} \arrow[l, "\mu_1"'] & \begin{matrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ (\frac{1+y_2x_1}{x_2}, \frac{y_1y_2x_1+y_1+x_2}{x_1x_2}) \\ (y_2^{-1}, y_1^{-1})\end{matrix} \arrow[l, "\mu_2"']
\end{tikzcd}$$

Figure 2.1 Cluster seeds along the mutation sequence $\mu_1\mu_2\mu_1\mu_2\mu_1$

Let $\Sigma_{t'} = \mu_1\mu_2\mu_1\Sigma_{t_o}$ and let $\Sigma_t = \mu_1\mu_2\Sigma_{t_o}$. Figure 2.1 says that $x_{1;t'} = \frac{1+y_2x_1}{x_2}$ and $y_{2;t'} = y_1^{-1}$, which we will calculate explicitly here by applying μ_1 to Σ_t. Recall that

$$x_{k;t'} = \frac{y_{k;t} \prod_j x_{j;t}^{[b_{jk}^t]_+} + \prod_j x_{j;t}^{[-b_{jk}^t]_+}}{(y_{k;t} \oplus 1)x_{k;t}}.$$

In this case, we have $k = 1$. Since the first column of the exchange matrix B_t has no positive entries,

$$y_{k;t} \prod_j x_{j;t}^{[b^t_{jk}]_+} = y_{1;t} = y_2$$

and

$$\prod_j x_{j;t}^{[-b^t_{jk}]_+} = x_{2;t} = \frac{y_1y_2x_1+y_1+x_2}{x_1x_2}.$$

Since $y_{1;t} = y_2$, we have $y_{k;t} \oplus 1 = 1$. Therefore the exchange relation says that

$$x_{1;t}x_{1;t'} = y_2 + \frac{y_1y_2x_1+y_1+x_2}{x_1x_2} = \frac{y_2x_1x_2+y_1y_2x_1+y_1+x_2}{x_1x_2} = \frac{(1+y_2x_1)(y_1+x_2)}{x_1x_2}.$$

Since $x_{1;t} = \frac{y_1+x_2}{x_1}$, indeed, we find that $x_{1;t'} = \frac{1+y_2x_1}{x_2}$. Since $k \neq 2$, to compute $y_{2;t'}$, we apply the mutation rule

$$y_{j;t'} = y_{j;t} y_{k;t'}^{[b^t_{kj}]_+} (y_{k;t} \oplus 1)^{-b^t_{kj}},$$

which specializes to

$$y_{2;t'} = y_{2;t} y_{1;t}^{[b^t_{12}]_+} (y_{1;t} \oplus 1)^{-b^t_{12}}.$$

Since $b^t_{12} = 1$, $y_{1;t} = y_2$, and $y_{2;t} = y_1^{-1}y_2^{-1}$, we have

$$y_{2;t'} = y_1^{-1}y_2^{-1}y_2 = y_1^{-1}.$$

Notice that the cluster seed at the end of the mutation sequence pictured in Figure 2.1 agrees with the initial cluster seed up to relabeling the cluster and coefficient variables and permuting the exchange matrix accordingly. By symmetry, applying μ_2 to Σ_{t_o} produces the second cluster seed in the second row up to permutation. Therefore, all the cluster variables of this cluster algebra appear in Figure 2.1. The cluster algebra is the subalgebra of $\mathbb{F}$ generated by the Laurent polynomials

$$x_1, x_2, \frac{y_1+x_2}{x_1}, \frac{y_1y_2x_1+y_1+x_2}{x_1x_2}, \frac{1+y_2x_1}{x_2}.$$

It is not always the case that the number of cluster variables is finite. The following is an example of a cluster algebra with countably many cluster variables.

Example 2.1.6 (The Kronecker cluster algebra)**.** The Kronecker cluster algebra is the cluster algebra defined by the initial exchange matrix

$$B_o = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

(In the next section, we will address how the Kronecker cluster algebra can also be defined by the Kronecker quiver K_2, as mentioned in the Introduction.) Let μ_+ denote the infinite mutation sequence $\mu_1\mu_2\mu_1\mu_2\ldots$, and let μ_- be $\mu_2\mu_1\mu_2\mu_1\ldots$. Let $\mu_+(0) = \mu_-(0)$ be the identity, and for $m > 0$, let $\mu_+(m)$ and $\mu_-(m)$ denote the sequences of the first m mutations in μ_+ and μ_- respectively. Since $\mu_1^2 = \mu_2^2 = 1$, every mutation sequence can be expressed as $\mu_+(m)$ or $\mu_-(m)$ for some $m \geq 0$.

Name the cluster variables as follows: for $m > 0$, let x_{m+2} be the last cluster variable obtained by applying the first m mutations of the sequence $\mu_1\mu_2\mu_1\mu_2\ldots$, and let x_{-m+1} be the last cluster variable obtained by applying the first m mutations of the sequence $\mu_2\mu_1\mu_2\mu_1\ldots$. By Theorem 4.1 in Caldero and Zelevinsky (2006), see also Theorem 2 in Musiker and Propp (2006) for a combinatorial approach, for $m > 0$,

$$x_{m+2} = x_1^{-m}x_2^{-m+1} \sum_{0\leq N\leq M\leq m} \binom{m-N}{m-M}\binom{M-1}{N} x_1^{2N}x_2^{2(m-M)}y_1^M y_2^N. \tag{2.1}$$

$$x_{-m+1} = x_1^{-m+1}x_2^{-m} \sum_{0\leq N\leq M\leq m} \binom{m-M}{m-N}\binom{M-1}{N} x_1^{2N}x_2^{2(m-M)}y_1^{m-1-N} y_2^{m-M}. \tag{2.2}$$

The reader might verify that Equation 2.1 indeed produces x_3 and x_4 as given below:

$$\begin{aligned} x_3 &= \frac{x_2^2 + y_1}{x_1}, \\ x_4 &= \frac{x_3^2 + y_1^2 y_2}{x_2} = \frac{(\frac{x_2^2+y_1}{x_1})^2 + y_1^2 y_2}{x_2} = \frac{x_2^4 + 2x_2^2 y_1 + y_1^2 + x_1^2 y_1^2 y_2}{x_1^2 x_2}. \end{aligned}$$

One may also verify using this explicit formula for the cluster variables x_n that x_n is different for each $n \in \mathbb{Z}$. Hence, the Kronecker cluster algebra has countably many cluster variables (compare with Example 2.1.5).

Since each coefficient variable $y_{i;t} \in \mathbb{P}$ is a Laurent monomial, it can be uniquely determined by its exponent vector. This leads to an alternative way of encoding the coefficient dynamics in our cluster seeds independently of the addition in coefficient semifield. We omit the tuple $\mathbf{y}$ of coefficient variables and extend the exchange matrix so that it is 2n-by-n. The columns of the bottom square matrix will correspond to the exponent vectors of the coefficient variables: $y_{i;t} = \prod_{j=1}^{n} y_j^{b_{n+j,i}}$.

Using this convention, a cluster seed is a tuple $\Sigma_t = (\mathbf{x}, \widetilde{B_t})$ where $\widetilde{B_t}$ is a 2n-by-n matrix with integer entries such that the top n-by-n part of $\widetilde{B_t}$ is skew-symmetrizable.

Definition 2.1.7 (Seed Mutation; Tall Matrix Version)**.** Mutating a cluster seed $\Sigma_t = (\mathbf{x}, \widetilde{B_t})$ gives a cluster seed $\Sigma_{t'} = (\mathbf{x}', \widetilde{B_{t'}})$ defined as follows:

- $\widetilde{B}_{t'} = (b'_{ij})$, where the mutation rule for each entry is the same as in Definition 2.1.3;
- $\mathbf{x}' = (x_{1;t'}, x_{2;t'}, \dots, x_{n;t'})$ where $x_{i;t'} = x_{i;t}$ if $i \neq k$ and

$$x_{k;t'} = \frac{\prod_j y_j^{[b_{n+j,k}]_+} \prod_j x_{j;t}^{[b_{jk}]_+} + \prod_j y_j^{[-b_{n+j,k}]_+} \prod_j x_{j;t}^{[-b_{jk}]_+}}{x_{k;t}}. \tag{2.3}$$

The rules for mutating the top part of $\widetilde{B}$ agree in the two definitions by construction. For the bottom part, since $y_{k;t'} = y_{k;t}^{-1}$, indeed we must have $b'_{n+i,k} = -b_{n+i,k}$. When $j \neq k$, the rule $y_{j;t'} = y_{j;t} y_{k;t}^{[b_{kj}]_+} (y_{k;t} \oplus 1)^{-b_{kj}}$ implies that

$$\begin{aligned} b'_{n+i,j} &= b_{n+i,j} + b_{n+i,k}[b_{kj}]_+ - \min(b_{n+i,k}, 0) b_{kj} \\ &= b_{n+i,j} + b_{n+i,k}[b_{kj}]_+ + [-b_{n+i,k}]_+ b_{kj} \\ &= \begin{cases} b_{n+i,j} & \text{if } b_{n+i,k} b_{kj} \leq 0, \\ b_{n+i,j} + b_{n+i,k} b_{kj} & \text{if } b_{n+i,k}, b_{kj} > 0, \\ b_{n+i,j} - b_{n+i,k} b_{kj} & \text{if } b_{n+i,k}, b_{kj} < 0, \end{cases} \end{aligned}$$

which agrees with this definition.

Example 2.1.8. The extended extended matrices for the cluster seeds in Figure 2.1 are as follows:

$$\begin{tikzcd}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \arrow[r, "\mu_1"] & \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \arrow[r, "\mu_2"] & \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \arrow[d, "\mu_1"] \\
\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \arrow[l, "\mu_1"'] & \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \arrow[l, "\mu_2"']
\end{tikzcd}$$

Figure 2.2 Mutations of extended exchange matrices for cluster seeds shown in Figure 2.1

In many situations, we would like to identify cluster seeds that are the same up to relabeling the columns and rows of its exchange matrices and relabeling $\mathbf{x}$ and $\mathbf{y}$ accordingly. For example, as discussed in Example 2.1.5, we might like to identify the first seeds on the two rows in the diagram of Figure 2.1. We use the term unlabeled seed to refer to a cluster seed up to such identification.

To better visualize the exchange relations of a cluster algebra, we often consider the graph whose the vertices are distinct unlabeled cluster seeds of a cluster algebra, where two vertices are connected by an edge labeled by elements of $[n]$ if and only if the corresponding seeds differ by a mutation in that direction. This is called the *exchange graph* of the cluster algebra. For example, the exchange graph for the cluster algebra of Example 2.1.5 is a cycle with five vertices and the exchange graph for the Kronecker cluster algebra is an infinite line with countably many vertices.

2.2 Skew-Symmetric Cluster Algebras

We say that the cluster algebra is *skew-symmetric* if its exchange matrices are skew-symmetric. When an exchange matrix of a cluster algebra is skew-symmetric, we may understand it as the signed incidence matrix of a quiver Q with n vertices $\{1, \ldots, n\}$. For $i, j \in [n]$, if $b_{ij} > 0$, we draw b_{ij} edges from vertex i to vertex j in the quiver. We may also understand the extended exchange matrix as the signed incidence matrix of a quiver with $2n$ vertices labeled $\{1, \ldots, n, 1', \ldots, n'\}$. The process of adding the n primed vertices is sometimes called framing the original quiver and the framed quiver is often written $\widetilde{Q}$ if the original quiver is Q. In addition to the edges of Q, if $b_{n+i,j} > 0$, we draw $b_{n+i,j}$ edges from vertex i' to vertex j, and $-b_{n+i,j}$ edges from vertex j to vertex i' otherwise. When the cluster algebra has principal coefficients, the bottom half of the extended exchange matrix of the initial cluster seed is the identity matrix. This means that framing the initial quiver corresponds to adding n primed vertices and adding one edge from j' to j for all $j \in [n]$.

Mutations can then be visualized as mutations of the corresponding quiver.

Definition 2.2.1 (Mutation at vertex k)**.** Given a seed $\Sigma_t = (\mathbf{x}, \mathbf{y}, B_t) = (\mathbf{x}, \widetilde{B_t})$ such that $\mathbf{x} = (x_{1;t}, \dots x_{n;t})$, $\mathbf{y} = (y_{1;t}, \dots y_{n;t})$, and $\widetilde{B_t}$ defines a framed quiver $\widetilde{Q}$ with vertices $\{1, \dots, n, 1', \dots, n'\}$, mutation at vertex k consists of the following steps:

1. For every path $i \to k \to j$, draw an edge $i \to j$;
2. Reverse the direction of all edges incident to k;
3. Delete all 2-cycles;
4. Update the cluster variable at vertex k to be

$$x_{k;t'} = \frac{\prod_{j \to k} x_{j;t} \prod_{j' \to k} y_{j;t} + \prod_{k \to j} x_{j;t} \prod_{k \to j'} y_{j;t}}{x_{k;t}}.$$

Note that in step 4, the product notation means that if there are two edges from j to k in $\widetilde{Q}$, then $x_{j;t}$ is multiplied twice in the product. The fact that this definition agrees with the mutation rule in Definition 2.1.3 has the immediate corollary that if the exchange matrix of any cluster seed is skew-symmetric, then exchange matrices at all seeds will be skew-symmetric.

Example 2.2.2. The two cluster algebras from Examples 2.1.5 and 2.1.6 are both skew-symmetric. Their corresponding quiver belongs to the family of r-Kronecker quivers, which are quivers with two vertices and r arrows from vertex 1 to vertex 2. When $r = 2$, the quiver is also just called the Kronecker quiver. When $r = 1$, this quiver is sometimes called the quiver of type A_2 because its underlying graph is the Dynkin diagram of the root system of type A_2.

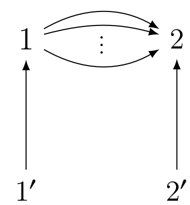


Figure 2.3 The framed r-Kronecker quiver

The following figure shows the mutation process for the 3-Kronecker quiver under the mutation sequence $\mu_1\mu_2\mu_1$. We label an edge $i \to j$ with the number of such edges if there are multiple edges from i to j.

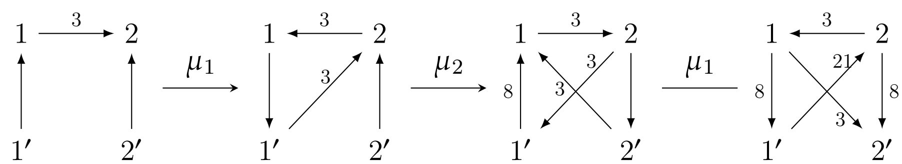


Figure 2.4 3-Kronecker quiver under mutations $\mu_1\mu_2\mu_1$

2.3 Parameterizations of Cluster Variables

To a cluster variable, we can associate a **d**-vector, a **c**-vector, a **g**-vector, and an F-polynomial. The **d** stands for denominator, **c** stands for coefficient, and **g** stands for grading. The F in F-polynomials stands for Fibonacci, but this is a slightly longer story. Fomin and Zelevinsky (2003b) studied certain Fibonacci polynomials, which are named so because in the case of bipartite type A cluster algebras, F-polynomials are a sum of a Fibonacci number of monomials (see for example the remark on page 5 and the discussion in Section 2.4 of Fomin and Zelevinsky (2003b)). But it was in Fomin and Zelevinsky (2007) that F-polynomials were introduced and defined as they are usually defined today.

These parameterizations are motivated by the fact that the recurrence of cluster variables is hard to solve. As we will see in the next section, these parameterizations motivate us to associate a cluster algebra with a root system. When we introduce Gupta's formula in the next section, we will also see that there are explicit expressions for cluster variables using only **c**- and **g**-vectors and the initial exchange matrix. The **c**-vectors and **g**-vectors are also deeply related to scattering diagrams, which will be introduced later, and they have representation-theoretic significance.

Given a vector with integer entries $\mathbf{c} = (c_1, \ldots, c_n)$, we often use the notation $\mathbf{x}^{\mathbf{c}} = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}$ and $\mathbf{y}^{\mathbf{c}} = y_1^{c_1} y_2^{c_2} \cdots y_n^{c_n}$.

Definition 2.3.1 ($\mathbf{c}$-, $\mathbf{d}$-vector)**.** A cluster variable $x_{i;t}$ can be written uniquely as $x_{i;t} = \mathbf{x}^{-d_{i;t}} p(\mathbf{x})$ so that $p(\mathbf{x})$ is some polynomial in $x_1, \ldots, x_n$ not divisible by x_i for any i. We call the vector $d_{i;t}$ the $\mathbf{d}$-vector of $\mathbf{x}_{i;t}$. The coefficient variable $y_{i;t}$ can be written uniquely as $y_{i;t} = \mathbf{y}^{\mathbf{c}_{i;t}}$, and $\mathbf{c}_{i;t}$ is the $\mathbf{c}$-vector of $\mathbf{x}_{i;t}$. In other words, $\mathbf{c}_{i;t}$ is the i-th column of the bottom half of the extended exchange matrix at the seed t. We write $\mathbf{c}_{i;t} = (\mathbf{c}_{i1}(t), \ldots, \mathbf{c}_{in}(t))$.

Recall that if $\Sigma_{t'}$ is obtained by mutating Σ_t in direction k, we have $y_{k;t'} = y_{k;t}^{-1}$, and $y_{j;t'} = y_{j;t} y_{k;t}^{[b_{kj}]_+} (y_{k;t} \oplus 1)^{-b_{kj}}$ for $j \neq k$. Given a vector $\mathbf{c}$, let $[\mathbf{c}]_+ = ([\mathbf{c}_1]_+, \ldots, [\mathbf{c}_n]_+)$. Then in the language of $\mathbf{c}$-vectors, we have the following recurrence:

$$\mathbf{c}_{j;t'} = \mathbf{c}_{j;t} + [b_{kj}]_+ \mathbf{c}_{k;t} - b_{kj} [-\mathbf{c}_{k;t}]_+. \tag{2.4}$$

The following seemingly elementary phenomenon of $\mathbf{c}$-vectors was first conjectured by Fomin and Zelevinsky (2007) and later proven in full generality in Gross et al. (2018) using the machinery of scattering diagrams.

Theorem 2.3.2 (Sign-coherence of $\mathbf{c}$-vectors, Gross et al. (2018))**.** The entries of any $\mathbf{c}$-vector have the same signs.

In other words, $[\mathbf{c}_{k;t}]_+ = \mathbf{0}$ or $[\mathbf{c}_{k;t}]_+ = \mathbf{c}_{k;t}$ for any $\mathbf{c}$-vector $\mathbf{c}_{k;t}$. We can easily check this theorem if we have the extended exchange matrix by looking at whether each column of the bottom square matrix contains entries with the same sign; see for instance, Figure 2.2.

To introduce $\mathbf{g}$-vectors, we first need to introduce a $\mathbb{Z}^n$-grading on $\mathbb{QP}(x_1, \ldots, x_n)$. Given a cluster algebra with the initial exchange matrix B_o, we let $\deg(x_i) = \mathbf{e}_i$ and $\deg(y_i) = -\mathbf{b}_{i;t_o}$, where $\mathbf{e}_i$ is the standard basis vector with a 1 at the i-th entry and 0's at other entries.

Proposition 2.3.3 (Proposition 6.1, Fomin and Zelevinsky (2007))**.** Every cluster variable is homogeneous with respect to this grading.

This proposition warrants the following definition.

Definition 2.3.4 ($\mathbf{g}$-Vector)**.** The $\mathbf{g}$-vector associated with a cluster variable $x_{i;t}$ is $\deg(x_{i;t})$.

From Equation 2.3 and Proposition 2.3.3, we can derive the following recurrence of $\mathbf{g}$-vectors.

Proposition 2.3.5 (Proposition 6.6, Fomin and Zelevinsky (2007))**.** Let $t', t \in \mathbb{T}_n$ be connected by an edge labeled k. Then $\mathbf{g}_{i;t'} = \mathbf{g}_{i;t}$ if $i \neq k$ and

$$\begin{aligned}\mathbf{g}_{k;t'} &= -\mathbf{g}_{k;t} - \sum_{j=1}^{n}[\mathbf{c}_{jk}(t)]_+\mathbf{b}_{j;t_o} + \sum_{j=1}^{n}[b_{jk}(t)]_+\mathbf{g}_{j;t} \\ &= -\mathbf{g}_{k;t} - \sum_{j=1}^{n}[-\mathbf{c}_{jk}(t)]_+\mathbf{b}_{j;t_o} + \sum_{j=1}^{n}[-b_{jk}]_+\mathbf{g}_{j;t}.\end{aligned} \tag{2.5}$$

By sign-coherence, depending on whether $\mathbf{c}_k$ is positive or negative, one of $\sum_{j=1}^{n}[\mathbf{c}_{jk}(t)]_+\mathbf{b}_{j;t_o}$ and $\sum_{j=1}^{n}[-\mathbf{c}_{jk}(t)]_+\mathbf{b}_{j;t_o}$ in fact vanishes to simplify the recurrence relation. Recurrences like this give us a way to understand the mutations of parameterizations of cluster variables without actually computing the cluster variables.

We will now compute some of the $\mathbf{c}$- and $\mathbf{g}$-vectors for the two cluster algebras in Examples 2.1.5 and 2.1.6.

Example 2.3.6. Let t_1 be the initial seed of the Kronecker cluster algebra, and let

$$t_{1+n} = \begin{cases} \mu_+(n)t_1 & \text{if } n \geq 0 \\ \mu_-(-n)t_1 & \text{if } n < 0 \end{cases}.$$

For $n > 1$, let i_n be such that $\mu_{i_n}t_{n-1} = t_n$; for $n < 1$, let i_n be such that $\mu_{i_n}t_{n+1} = t_n$.

Then under the numbering of cluster variables in Example 2.1.6, we have $\{x_n, x_{n+1}\}$ as the two cluster variables in the cluster that corresponds to t_n.

For $n \neq 1, 2$, let

$$\mathbf{c}_n = \begin{cases} \mathbf{c}_{i_{n-1};t_{n-1}} & \text{if } n > 2, \\ \mathbf{c}_{i_n;t_n} & \text{if } n < 1, \\ \mathbf{c}_{n;t_1} & \text{if } n = 1, 2. \end{cases}$$

Define $\mathbf{g}_n, \mathbf{d}_n$ analogously.

Note that the vectors $\mathbf{g}_n, \mathbf{d}_n$ can be equivalently defined as the $\mathbf{g}$, $\mathbf{d}$-vectors of the cluster variable x_n. Among all seeds t_n, there are two adjacent seeds whose clusters contain a certain x_n, and so it is not well-defined to speak of the $\mathbf{c}$-vector that correspond to x_n. Our definition above formalizes the idea that $\mathbf{c}_n$ should denote the $\mathbf{c}$-vector of x_n at the cluster which is closer to t_1.

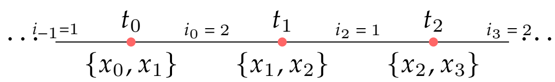


Figure 2.5 The exchange graph $\mathbb{T}_2$ of an infinite rank-two cluster algebra

Using Equation 2.1 and Equation 2.2, we see that for $m \geq 1$,

$$\mathbf{d}_{m+2} = \begin{bmatrix} m \\ m-1 \end{bmatrix}, \ \mathbf{d}_{-m+1} = \begin{bmatrix} m-1 \\ m \end{bmatrix}.$$

For $m \geq 1$, we can calculate $\mathbf{g}_{m+2}$ by considering the monomial that corresponds to $M = 0$, $N = 0$, namely $x_1^{-m} x_2^{-m+1} x_2^{2m}$, which has degree $\mathbf{g}_{m+2} = \begin{bmatrix} -m \\ m+1 \end{bmatrix}$. Similarly, we can calculate $\mathbf{g}_{-m+1}$ by considering the monomial that corresponds to $M = m$, $N = m-1$, namely $x_1^{-m+1} x_2^{-m} x_1^{2(m-1)}$, which has degree $\mathbf{g}_{-m+1} = \begin{bmatrix} m-1 \\ -m \end{bmatrix}$. Lastly, for $m \geq 1$,

$$\mathbf{c}_{m+2} = \begin{bmatrix} -m \\ -m+1 \end{bmatrix}, \ \mathbf{c}_{-m} = \begin{bmatrix} m-2 \\ m-1 \end{bmatrix}.$$

The $\mathbf{c}$, $\mathbf{g}$-vectors are related by what's called tropical duality.

Theorem 2.3.7 (Theorem 1.2, Nakanishi and Zelevinsky (2012))**.** Let G_t^B and C_t^B be the matrices whose i-th columns are $\mathbf{g}_{i;t}$ and $\mathbf{c}_{i;t}$ respectively in the cluster algebra with initial exchange matrix B. Then

$$(G_t^B)^T = (C_t^{-B^T})^{-1}.$$

Equivalently, if we write $\tilde{\mathbf{c}}_{i;t}$ for the i-th column of $C_t^{-B^T}$, then

$$\mathbf{g}_{i;t} \cdot \tilde{\mathbf{c}}_{j;t} = \delta_{ij}.$$

We may understand a cluster variable $x_{i;t} \in \mathbb{QP}(x_1, \ldots, x_n)$ as a rational function in $x_1, \ldots, x_n, y_1, \ldots, y_n$. When we understand it as a function, we write it as $X_{i;t}(x_1, \ldots, x_n, y_1, \ldots, y_n)$. In the foundational paper Fomin and Zelevinsky (2002) on cluster algebras, Fomin and Zelevinsky establish a property of $X_{i;t}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ called the Laurent phenomenon, which is sharpened in Fomin and Zelevinsky (2003a) for the principal coefficients case.

Theorem 2.3.8 (Proposition 11.2, Fomin and Zelevinsky (2003a), as cited in Proposition 3.6 of Fomin and Zelevinsky (2007))**.** Let $\mathcal{A}$ be a cluster algebra with principal coefficients at the initial seed. Then $\mathcal{A} \subset \mathbb{Z}[\mathbf{x}^{\pm 1}; \mathbf{y}]$. That is, every element of $\mathcal{A}$ is a Laurent polynomial in $x_1, \dots, x_n$ whose coefficients are integer polynomials in $y_1, \dots, y_n$.

As a result, when we specialize to let $x_i = 1$ for all $i \in [n]$, we obtain a polynomial.

Definition 2.3.9 (F-polynomials)**.** Given a cluster variable $x_{i;t}$, the associated F-polynomial is

$$F_{i;t}(y_1, \dots, y_n) = X_{i;t}(1, 1, \dots, 1, y_1, \dots, y_n).$$

Using Proposition 2.3.3, Fomin and Zelevinsky (2007) derives the following corollary, which says that the $\mathbf{g}$-vector and F-polynomial together determine the cluster variable.

Corollary 2.3.10 (Corollary 6.3, Fomin and Zelevinsky (2007))**.** Let $\widehat{y}_i = y_i \mathbf{x}^{\mathbf{b}_{i;t_0}}$. Then

$$X_{i;t}(x_1, \dots, x_n, y_1, \dots, y_n) = \mathbf{x}^{\mathbf{g}_{i;t}} F_{i;t}(\widehat{y}_1, \dots, \widehat{y}_n).$$

Note how the monomials $\widehat{y}_i = y_i \mathbf{x}^{\mathbf{b}_{j;t_0}}$ have degree $\mathbf{0}$, which demonstrates that $X_{i;t}$ is homogeneous of degree $\mathbf{g}_{i;t}$.

We shall later need the following proposition about how the $\widehat{y}$'s mutate.

Proposition 2.3.11 (Proposition 3.9, Fomin and Zelevinsky (2007))**.** Let t and t' be related by μ_k. Then

$$\widehat{y}_{j;t'} = \begin{cases} \widehat{y}_{k;t}^{-1} & \text{if } j = k, \\ \widehat{y}_{j;t} \widehat{y}_{k;t}^{[b^t_{kj}]_+} (\widehat{y}_{k;t} + 1)^{-b^t_{kj}} & \text{otherwise.} \end{cases} \tag{2.6}$$

2.4 A Root-Theoretic Perspective

Since root systems were discovered and used for the classification of semisimple Lie algebras, its ubiquity has been demonstrated in many areas. Amazingly, cluster algebras turn out to also be deeply related to root systems.

A rank-n (crystallographic) root system Δ consists a set of integer-valued nonzero vectors in $\mathbb{R}^n$ that we call *roots*. We will not define root systems rigorously here, but we list some of its properties and associated definitions.

One way to specify a root system is with a n-by-n matrix $A = (a_{ij})$ such that $a_{ii} = 2$ for all $i \in [n]$, $a_{ij} \in \mathbb{Z}_{\leq 0}$, and $a_{ij} = 0$ implies $a_{ji} = 0$. Such a matrix is called a *Cartan matrix*. The Cartan matrix specifies the angles that the roots need to be at with respect to each other. A Cartan matrix is *indecomposable* if there is no way to relabel the rows and columns so that the matrix is in block diagonal form with at least two blocks. Indecomposable Cartan matrices can be completely classified into three types.

Theorem 2.4.1 (Theorem 4.3 and Corollary 4.3, Kac (1990))**.** Let A be an indecomposable Cartan matrix. Then one and only one of the following three possibilities holds for both A and A^T:

- (*finite type*) $\det A \neq 0$; there exists $u > 0$ such that $Au > 0$; $Av > 0$ implies $v > 0$ or $v = 0$;
- (*affine type*) $\operatorname{corank} A = 1$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$;
- (*indefinite type*) there exists $u > 0$ such that $Au < 0$; $Av \geq 0, v \geq 0$ imply $v = 0$.

So A is of finite (resp. affine or indefinite) type if and only if there exists $\alpha > 0$ such that $A\alpha > 0$ (resp. $= 0$ or < 0).

For the scope of this thesis, it is most useful for us to know more about root systems of affine type. The following lemma is a useful fact.

Lemma 2.4.2 (Lemma 4.6, Kac (1990))**.** Let $A = (a_{ij})$ be an indecomposable Cartan matrix of finite or affine type. Then A is symmetrizable.

Since A is symmetrizable, there exists δ_i such that for $i \neq j$, $\delta_i b_{ij} = \delta_j b_{ji}$. Since all entries take integer value, we may take δ_i so that $\delta_i^{-1} \in \mathbb{Z}$ for all $i \in [n]$ and $\gcd(\delta_1^{-1}, \ldots, \delta_n^{-1}) = 1$.

There exists a set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of n *simple roots* such that every root is a linear combination with all non-positive or all non-negative integer coefficients. We write Δ_+ and Δ_- to denote the *positive* and *negative* roots, which are non-negative and non-positive linear combinations of the simple roots respectively. Then $\Delta = \Delta_+ \cup \Delta_-$. Let $\alpha_i^\vee = \delta_i^{-1}\alpha_i$ and for $\alpha \in \Delta$, define $\alpha^\vee$ by extending this linearly. If α is a root, then $\alpha^\vee$ is called a *coroot*. Let $Q = \operatorname{span}(\alpha_1, \ldots, \alpha_n)$ be the *root lattice* and let $Q^\vee = \operatorname{span}(\alpha_1^\vee, \ldots, \alpha_n^\vee)$ be the *coroot lattice*. Note that elements of the root lattice are not necessarily roots.

Let $K(\cdot,\cdot) : Q^\vee \times Q \to \mathbb{Z}$ be the bilinear form defined by $K(\alpha_i^\vee, \alpha_j) = a_{ij}$. Then because of the axioms for root systems that we avoided talking about, the reflection s_α across the hyperplane normal to α corresponds to $s_\alpha(\beta) = \beta - K(\alpha^\vee, \beta)\alpha$. We write $s_i = s_{\alpha_i}$ for the *simple reflection* at the simple root α_i. A key property of the root system Δ is that it is closed under reflection across hyperplanes normal to any root. The group of reflections generated by simple reflections $W = \langle s_i : i \in [n]\rangle$ is called the *Coxeter group*.

Roots are either real or imaginary. Let Δ^{re} and Δ^{im} be the set of real and imaginary roots respectively, then $\Delta = \Delta^{\text{re}} \cup \Delta^{\text{im}}$. The general definition for real and imaginary roots does not concern us here, but the following characterization are useful to have for the finite and affine cases.

Theorem 2.4.3 (Theorem 5.6, Kac (1990))**.** Let A be an indecomposable generalized Cartan matrix.

1. If A is of finite type, then the set Δ^{im} is empty.
2. If A is of affine type, then

$$\Delta^{\text{im}} = \{n\delta : n \in \mathbb{Z}\},$$

   where δ is such that $A\delta = 0$, $\delta > 0$ and it is closest to 0 among all such vectors.

Moreover, in the affine case, there exists aff $\in [n]$ such that the Cartan matrix restricted to rows and columns $[n]\backslash\text{aff}$ produces a root system $\Delta_{\text{fin}} \subset \Delta$ of finite type with the property that $\Delta^{\text{re}} = \{\alpha + n\delta : \alpha \in \Delta_{\text{fin}}, n \in \mathbb{Z}\}$.

We are now ready to look at an example of the relationship between cluster algebras and root systems.

Example 2.4.4 (The Kronecker cluster algebra and the root system $A_1^{(1)}$)**.** Recall our calculations for the **d**-vectors $\mathbf{d}_m$ of the Kronecker cluster algebra in Example 2.3.6. The set of **d**-vectors is exactly

$$\left\{\begin{bmatrix}1\\0\end{bmatrix} + n\begin{bmatrix}1\\1\end{bmatrix} : n \in \mathbb{Z}_{\geq 0}\right\} \bigcup \left\{\begin{bmatrix}0\\1\end{bmatrix} + n\begin{bmatrix}1\\1\end{bmatrix} : n \in \mathbb{Z}_{\geq 0}\right\} \cup \left\{\begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}\right\}.$$

As m tends to $\pm\infty$, $\mathbf{d}_m$ approaches but never reaches the direction $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$.

On the other hand, the affine root system $\widetilde{\Delta_1}$ of type $A_1^{(1)}$ has two simple roots α_1 and α_2 and an imaginary root $\delta = \alpha_1 + \alpha_2$. The set

$$\{\alpha_1 + n\delta : n \in \mathbb{Z}_{\geq 0}\} \cup \{\alpha_2 + n\delta : n \in \mathbb{Z}_{\geq 0}\}$$

is exactly the set of positive real roots in $\widetilde{\Delta_1}$. So the set of **d**-vectors differs from the set of positive real roots by the two negative simple roots.

Our calculation also shows that the set of $\mathbf{c}_m$'s as well as their negations, which together is the complete set of **c**-vectors of the Kronecker cluster algebra, coincides with real roots of $A_1^{(1)}$. There is also a connection between **g**-vectors of the Kronecker cluster algebra with the root system $\widetilde{\Delta_1}$, which is slightly harder to state.

There are some other clues that suggest we might look at the affine root system of type $A_1^{(1)}$ alongside the Kronecker cluster algebra. It turns out that the underlying undirected graph of the Kronecker quiver is the same as the Dynkin diagram of type $A_1^{(1)}$. In addition, the Cartan matrix of type $A_1^{(1)}$ is

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix},$$

which can be obtained from the initial exchange matrix of the Kronecker cluster algebra

$$B_o = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

by making all entries negative and letting the diagonal entries be 2.

This is an example of a general correspondence. One of the first major achievements of the study of cluster algebras is the classification of cluster algebras of finite type. A cluster algebra is of *finite type* if it has finitely many clusters. Given an $n \times n$ exchange matrix B of a cluster algebra, we can associate it with a weighted directed graph $\Gamma(B)$ with n vertices, where there is an edge from vertex i to vertex j if and only if $b_{ij} > 0$, in which case we give it a weight of $|b_{ij}b_{ji}|$.

Theorem 2.4.5 (Theorem 1.8, Fomin and Zelevinsky (2003a), as cited in Theorem 2.34 of Williams (2014))**.** The cluster algebra A is of finite type if and only if it has a seed $(B, \mathbf{x}, \mathbf{y})$ such that $\Gamma(B)$ is an orientation of a finite type Dynkin diagram.

This correspondence agrees with a different way of connecting cluster algebras and root systems. Given an exchange matrix B, consider its *Cartan companian* $A(B)$ defined by letting all diagonal entries be 2 and modifying its off-diagonal entries to be non-positive. Since B is skew-symmetrizable, $A(B)$ will be symmetrizable, which is what we expect based on Lemma 2.4.2. Moreover, Fomin and Zelevinsky (2003a) showed that the correspondence between *almost positive roots* (the positive roots and $-\Pi$) and cluster variables that we saw in Example 2.4.4 holds in general for finite-type cluster algebras.

Theorem 2.4.6 (Theorem 1.9, Fomin and Zelevinsky (2003a))**.** Let Δ be the root system that corresponds to the cluster algebra $\mathcal{A}$. There is a bijection between the almost positive roots in Δ and the cluster variables in $\mathcal{A}$ by sending an almost positive root α to the unique cluster variable whose **d**-vector is α written in the basis of simple roots.

This so-called almost positive roots model for finite type is extended to a uniform finite/affine model recently in Reading and Stella (2020). Prior to their work, there was also successful representation-theoretic generalizations of this classification to the affine and indefinite case.

2.5 Scattering Diagrams

Scattering diagrams come to the field of cluster algebras from algebraic geometry and mirror symmetry. In Gross et al. (2018), they were used to resolve numerous fundamental conjectures on cluster algebras, including the sign-coherence conjecture (Theorem 2.3.2) and the positivity conjecture (for the skew-symmetrizable case). They will also help motivate the limits that we consider in Chapter 5. In our introduction of scattering diagrams below, we will mostly be following the notation of Reading (2020b), which takes the perspective of root combinatorics.

Given an $n \times n$ exchange matrix $B = (b_{ij})$, let $A(B)$ be its Cartan companion. We will inherit all the notation related to the corresponding root system from the previous section with $A(B)$ as the defining data. Also let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denote the standard Euclidean inner product and let $w(\cdot, \cdot) : Q^\vee \times Q \to \mathbb{Z}$ be the bilinear form defined by $w(\alpha_i^\vee, \alpha_j) = b_{ij}$.

A *wall* $(\mathfrak{d}, f_\mathfrak{d})$ consists of a codimension-1 cone in $\mathbb{R}^n$ contained in $\beta^\perp$ for some $\beta \in Q^+$ and a formal power series $f_\mathfrak{d} = f_\mathfrak{d}(\widehat{y}^\beta) \in \Bbbk[[\widehat{y}^\beta]]$. A *scattering diagram* is a collection $\mathfrak{D}$ of walls with a certain finiteness condition. More precisely, let $\mathfrak{m}$ be the maximal ideal of $k[[\widehat{\mathbf{y}}]]$ that consists of all formal power series in $\widehat{y}_1, \ldots, \widehat{y}_n$ with a constant term of zero. For $k \geq 1$, let $\mathfrak{D}_k \subseteq \mathfrak{D}$ denote the scattering diagram whose walls $(\mathfrak{d}, f_\mathfrak{d})$ are such that $f_\mathfrak{d} \not\equiv 1$ modulo $\mathfrak{m}^{k+1}$. Then we require that there are only finitely many walls in $\mathfrak{D}_k$. This implies that there are only countably many walls in any scattering diagram.

Given a scattering diagram, a path $\gamma : [0, 1] \to \mathbb{R}^n$ is generic if it only crosses walls transversally in their relative interiors and have endpoints outside walls. A wall $(\mathfrak{d}, f_{\mathfrak{d}})$ of the scattering diagram and a generic path γ in $\mathbb{R}^n$ together define a *wall-crossing automorphism* $\mathfrak{p}_{\gamma,\mathfrak{d}} : \Bbbk[[\mathbf{x}, \widehat{\mathbf{y}}]] \to \Bbbk[[\mathbf{x}, \widehat{\mathbf{y}}]]$ given by

$$\mathfrak{p}_{\gamma,\mathfrak{d}}(\mathbf{x}^\lambda) = \mathbf{x}^\lambda f_{\mathfrak{d}}^{\langle \lambda, \pm\beta^\vee \rangle}$$
$$\mathfrak{p}_{\gamma,\mathfrak{d}}(\widehat{\mathbf{y}}^\phi) = \widehat{\mathbf{y}}^\phi f_{\mathfrak{d}}^{w(\pm\beta^\vee, \phi)}$$

where we choose $+$ if γ crosses against β and $-$ if γ crosses with β. Let $\mathfrak{p}_{\gamma,\mathfrak{D}_k}$ denote the composition of $\mathfrak{p}_{\gamma,\mathfrak{d}}$ for all $\mathfrak{d} \in \mathfrak{D}_k$ crossed by γ so that a wall-crossing automorphism is applied first if the wall is crossed by γ first. The path-ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}} : \Bbbk[[\mathbf{x}, \widehat{\mathbf{y}}]] \to \Bbbk[[\mathbf{x}, \widehat{\mathbf{y}}]]$ defined by a generic path $\gamma : [0, 1] \to \mathbb{R}^n$ is then $\lim_{k\to\infty} \mathfrak{p}_{\gamma,\mathfrak{D}_k}$. From the definition, we see that the path-ordered product is only sensitive to what walls are crossed by the path and the direction that the crossing happened in.

We say that a scattering diagram is *consistent* if the path-ordered product depends on only the start and end points of the path. Given an initial scattering diagram consisting of walls $\{(\alpha_i^\perp, 1 + \widehat{y}_i) : i \in [n]\}$ and the exchange matrix B, there exists a unique[1] scattering diagram $\mathrm{Scat}^T(B)$[2].

A lot is known about the geometric structure of the scattering diagram. Let $\mathrm{ScatFan}^T(B)$ be the fan in $\mathbb{R}^n$ whose codimension-1 skeleton is the scattering diagram $\mathrm{Scat}^T(B)$. Let the cone spanned by $\alpha_i^\perp$ for $i \in [n]$ be the dominant chamber and let $\mathrm{ChamberFan}^T(B)$ be the subfan in $\mathrm{ScatFan}^T(B)$ of maximal cones that can reached from the dominant chamber by a finite sequence of adjacent cones. Let $\mathbf{g}\mathrm{Fan}(B)$ be the set of all cones C that are the nonnegative linear span of $\mathbf{g}$-vectors that belong to the same cluster. We cite Reading (2020b) for the following result, but it follows a combination of results from previous research.

Theorem 2.5.1 (Corollary 2.6, Reading (2020b))**.** The set $\mathbf{g}\mathrm{Fan}(B)$ is a fan and coincides with $\mathrm{ChamberFan}^T(B)$.

While we don't know everything about a scattering diagram, this theorem grounds us with the intuition that maximal cones reachable from the dominant chamber by a finite sequence are exactly the clusters of the cluster algebra, and in terms of Euclidean coordinates, these cones are spanned by the $\mathbf{g}$-vectors of the cluster variables in the corresponding cluster, and two maximal cones

[1] Up to an equivalence made precise in Reading (2020b).

[2] The transpose superscript emphasizes a choice of convention explained in Reading (2020b) which that does not concern us here.

of $\text{ChamberFan}^T(B)$ are adjacent if and only if the corresponding clusters are related by a mutation.

The following result of Bridgeland (as cited in Reading (2020b)) shows us that root systems and scattering diagrams are very related.

Theorem 2.5.2 (Theorem 2.3, Reading (2020b))**.** If B is skew-symmetric and the associated quiver admits a genteel potential (and in particular, if B is acyclic), then every wall of $\text{Scat}^T(B)$ is normal to a positive root.

By the duality of $\mathbf{c}$ and $\mathbf{g}$-vectors, since $\mathbf{c}$-vectors are real roots, walls in $\mathbf{g}\text{Fan}(B)$ are normal to real roots. Thus, the following theorem tells us exactly what the wall function is when the wall is normal to a real root.

Theorem 2.5.3 (Theorem 4.6, Gross et al. (2018) as cited in Theorem 2.8 of Reading (2020b))**.** Let $\mathfrak{D} = \text{Scat}^T(B)$ and let $f_p = \prod_{\mathfrak{d} \ni p} f_{\mathfrak{d}}$. If F and G are adjacent maximal cones of $\mathbf{g}\text{Fan}(B)$, then $f_p(\mathfrak{D}) = 1 + \widehat{y}^\beta$ for every general point p in $F \cap G$, where β is the primitive root normal to $F \cap G$ in Q^+.

Theorem 2.5.4 (Theorem 5.6, Gross et al. (2018), as cited in Theorem 2.9 and Corollary 2.10 of Reading (2020b))**.** Let $\mathfrak{D} = \text{Scat}^T(B)$. If λ is contained in a maximal cone C of $\mathbf{g}\text{Fan}(B)$ and γ is a generic path such that $\gamma(0)$ lies in the interior of C and $\gamma(1)$ lies in the interior of the dominant chamber, then the cluster variable with $\mathbf{g}$-vector λ is $\mathfrak{p}_{\gamma,\mathfrak{D}}(\mathbf{x}^\lambda)$ and the F-polynomial is $\mathbf{x}^{-\lambda}\mathfrak{p}_{\gamma,\mathfrak{D}}(\mathbf{x}^\lambda)$.

This theorem is actually stated for *cluster monomials*, which are monomials in the cluster variables of some seed, but we don't need that level of generality here.

Example 2.5.5. Let's try an example! Let $\mathfrak{D}$ be the scattering diagram defined by the initial exchange matrix $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. The path γ labeled in Figure 2.6 crosses two walls. The first wall is normal to the positive root $2\alpha_1 + \alpha_2$ and the second wall is normal to α_1. So by Theorem 2.5.3, we know that the wall functions are $1 + \widehat{y}_1^2\widehat{y}_2$ and $1 + \widehat{y}_1$ respectively. At both walls, γ crosses the wall with the positive root, so we take the negative sign in the definition of the wall-crossing automorphism. Since the Kronecker quiver is skew-symmetric, $\delta_1 = \delta_2 = 1$ and $\alpha^\vee = \alpha$ so we don't have to worry about the $\vee$'s in the definition. By the definition of the bilinear form $w(\cdot,\cdot)$, explicitly, we have

$$w\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = 2(ad - bc).$$

By Theorem 2.5.4, we expect that if we apply $\mathfrak{p}_{\gamma,\mathfrak{D}}$ to $x_1^{-2}x_2^3$, we will obtain the cluster variable x_4. Crossing the first wall, we have

$$x_1^{-2}x_2^3 \mapsto x_1^{-2}x_2^3(1+\widehat{y}_1^2\widehat{y}_2)^{\langle \left[\begin{smallmatrix}-2\\3\end{smallmatrix}\right], -\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]\rangle} = x_1^{-2}x_2^3(1+\widehat{y}_1^2\widehat{y}_2).$$

For the second wall, we can compute the image of $x_1^{-2}x_2^3$ and $\widehat{y}_1^2\widehat{y}_2$ separately first. We get that

$$x_1^{-2}x_2^3 \mapsto x_1^{-2}x_2^3(1+\widehat{y}_1)^{\langle \left[\begin{smallmatrix}-2\\3\end{smallmatrix}\right], -\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\rangle} = x_1^{-2}x_2^3(1+\widehat{y}_1)^2$$

and

$$\widehat{y}_1^2\widehat{y}_2 \mapsto \widehat{y}_1^2\widehat{y}_2(1+\widehat{y}_1)^{w(-\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right])} = \widehat{y}_1^2\widehat{y}_2(1+\widehat{y}_1)^{-2}.$$

So

$$\mathfrak{p}_{\gamma,\mathfrak{D}}(x_1^{-2}x_2^3) = x_1^{-2}x_2^3(1+\widehat{y}_1)^2(1+\widehat{y}_1^2\widehat{y}_2(1+\widehat{y}_1)^{-2}) = x_1^{-2}x_2^3((1+\widehat{y}_1)^2+\widehat{y}_1^2\widehat{y}_2),$$

which is what we expect since

$$F_4(y_1, y_2) = (1+y_1)^2 + y_1^2y_2.$$

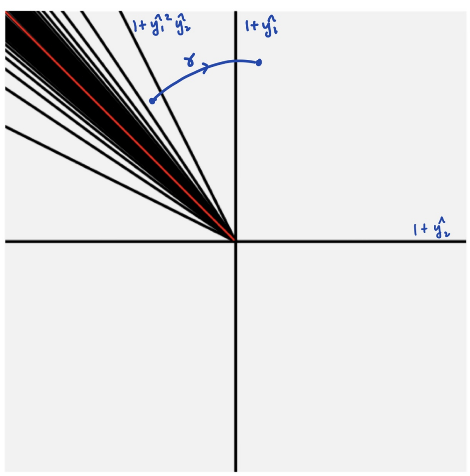


Figure 2.6 A path γ in the scattering diagram for the Kronecker quiver with the exchange matrix, figure from Page 17 of Reading (2018), my annotation

Chapter 3

Gupta's Formula

In Gupta (2018), Meghal Gupta introduces a formula for F-polynomials of all skew-symmetrizable cluster algebras in terms of **c**- and **g**-vectors and proves her formula using only elementary combinatorics. Due to Corollary 2.3.10, her formula can also be used to compute cluster variables.

This is a remarkable result for its wide generality, easily computable nature, and elementary method of proof. It should be mentioned that Gupta's formula is essentially not new, and can be obtained by specializing results of Gross et al. (2018) and Nagao (2013), which were proven with higher machinery, as we will discuss in Section 3.2. Yet the value of Gupta's perspective is that it provides a self-contained, completely combinatorial approach to this formula.

In Gupta (2018), Gupta's formula is not written directly in terms of **c**- and **g**-vectors, but in terms of certain $a_{i,j}, b_{i,j}$'s that are closely related to **c**- and **g**-vectors (see Definition 2.15 in Gupta (2018)). The slides Musiker (2019) translate Gupta's formula in the skew-symmetric case to its modern form, which uses **c**- and **g**-vectors more transparently. The slides Musiker (2019) also includes a mostly complete inductive proof for the skew-symmetric case of Gupta's formula, independently of Gupta (2018).

In Section 3.1, we document the work of Musiker (2019) in translating Gupta's work in full because to this day this work remain unpublished. We also generalize the statements and the proof in Musiker (2019) to skew-symmetrizable cluster algebras, which completes this work of translation from Gupta (2018). After the statement and proof of Gupta's formula, we provide an example of applying it to a non-skew-symmetric cluster algebra. In Section 3.2, we briefly discuss the connection between Gupta's formula and Nagao (2013). We also prove in detail how Gupta's formula is related to the scattering diagrams and path-ordered products of Gross et al. (2018).

3.1 Statement and Proof of Gupta's Formula

Theorem 3.1.1. Given a mutation sequence $\mu_{i_1}\mu_{i_2}\ldots$, let t_j be the seed obtained by applying the mutations $\mu_{i_1}\mu_{i_2}\ldots\mu_{i_j}$ to the initial seed in the cluster algebra defined by the exchange matrix B_o, and let $\tilde{t_j}$ be the analogous seed in the cluster algebra defined by the exchange matrix $-B_o^T$. Let $\mathbf{c}_j = \mathbf{c}_{i_j;t_j}$, $\mathbf{g}_j = \mathbf{g}_{i_j;t_j}$, and $\tilde{\mathbf{c}}_j = \mathbf{c}_{i_j;\tilde{t}_j}$. Then the ℓ-th F-polynomial along the mutation sequence is

$$F_{i_\ell;t_\ell}(\mathbf{y}) = \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j\cdot\mathbf{g}_\ell}\Big|_{z_i=\mathbf{y}^{|\mathbf{c}_i|}} \text{ where } L_1 = 1+z_1, L_k = 1+z_k\prod_{j=1}^{k-1} L_j^{\tilde{\mathbf{c}}_j\cdot B_o|\mathbf{c}_k|}.$$

Proof. To simplify the notation, we will understand the z_i's to be specialized as $z_i = \mathbf{y}^{|\mathbf{c}_i|}$ for the rest of the proof.

We shall prove the following formula for each F-polynomial at the seed t_ℓ, which specializes to the desired theorem when $i = i_\ell$:

$$F_{i;t_\ell} = \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j\cdot\mathbf{g}_{i;t_\ell}} \text{ where } L_1 = 1+z_1, L_k = 1+z_k\prod_{j=1}^{k-1} L_j^{\tilde{\mathbf{c}}_j\cdot B_o|\mathbf{c}_k|}.$$

We proceed by induction on ℓ. The base case is $\ell = 0$, where the formula above reduces to the empty product, which we interpret to be 1 by convention. Since we are at the initial seed, the F-polynomial for each cluster variable is just 1. So the F-polynomials at t_o agree with the formula.

Now suppose that the formula is correct for some $\ell \geq 0$, and let $k = i_{\ell+1}$, $t = t_\ell$ and $t' = t_{\ell+1}$. By Theorem 2.3.7, $\tilde{\mathbf{c}}_{\ell+1}\cdot g_{i;t'} = 0$ if $i \neq k$. We know that if $i \neq k$, the F-polynomial and the $\mathbf{g}$-vector at i do not change as we mutate from t to t'. Therefore, for $i \neq k$,

$$F_{i;t'} = F_{i;t} = \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j\cdot\mathbf{g}_{i;t}} = \prod_{j=1}^{\ell+1} L_j^{\tilde{\mathbf{c}}_j\cdot\mathbf{g}_{i;t'}}.$$

Now consider the F-polynomial at k. By the recurrence of F-polynomials, we know that

$$F_{k;t'} = \frac{\mathbf{y}^{[\mathbf{c}_{k;t}]_+}\prod_{i=1}^n F_{i;t}^{[\mathbf{b}_{ik}(t)]_+} + \mathbf{y}^{[-\mathbf{c}_{k;t}]_+}\prod_{i=1}^n F_{i;t}^{[-\mathbf{b}_{ik}(t)]_+}}{F_{k;t}}.$$

We can substitute $F_{i;t}$ in the numerator and $F_{k;t}$ in the denominator with products of L_j's using the inductive hypothesis. Notice that for each $F_{i;t}$, the exponents on L_j are always a dot product of $\tilde{\mathbf{c}}_j$ with some $\mathbf{g}$-vector. After consolidating the exponents and in particular factoring out $\tilde{\mathbf{c}}_j$, we get that

$$F_{k;t'} = \mathbf{y}^{[\mathbf{c}_{k;t}]_+} \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot (\sum_{i=1}^{n} [\mathbf{b}_{ik}(t)]_+ \mathbf{g}_{i;t} - \mathbf{g}_{k;t})} + \mathbf{y}^{[-\mathbf{c}_{k;t}]_+} \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot (\sum_{i=1}^{n} [-\mathbf{b}_{ik}(t)]_+ \mathbf{g}_{i;t} - \mathbf{g}_{k;t})},$$

where the $-\mathbf{g}_{k;t}$ comes from $F_{k;t}$ in the denominator.

By Proposition 2.3.5,

$$\sum_{i=1}^{n} [\mathbf{b}_{ik}(t)]_+ \mathbf{g}_{i;t} - \mathbf{g}_{k;t} = \mathbf{g}_{k;t'} + \sum_{j=1}^{n} [c_{jk}(t)]_+ \mathbf{b}_{j;t_o} = \mathbf{g}_{k;t'} + B_o [\mathbf{c}_{k;t}]_+$$

and

$$\sum_{i=1}^{n} [-\mathbf{b}_{ik}(t)]_+ \mathbf{g}_{i;t} - \mathbf{g}_{k;t} = \mathbf{g}_{k;t'} + \sum_{j=1}^{n} [-c_{jk}(t)]_+ \mathbf{b}_{j;t_o} = \mathbf{g}_{k;t'} + B_o [-\mathbf{c}_{k;t}]_+.$$

So

$$F_{k;t'} = \mathbf{y}^{[\mathbf{c}_{k;t}]_+} \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot (\mathbf{g}_{k;t'} + B_o [\mathbf{c}_{k;t}]_+)} + \mathbf{y}^{[-\mathbf{c}_{k;t}]_+} \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot (\mathbf{g}_{k;t'} + B_o [-\mathbf{c}_{k;t}]_+)}.$$

By sign-coherence (Theorem 2.3.2), either $c_{jk}(t) \geq 0$ for all $j = 1, \ldots n$, or $c_{jk}(t) \leq 0$ for all $j = 1, \ldots n$. In the first case,

$$F_{k;t'} = \mathbf{y}^{\mathbf{c}_{k;t}} \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot (\mathbf{g}_{k;t'} + B_o \mathbf{c}_{k;t})} + \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot \mathbf{g}_{k;t'}};$$

in the second case,

$$F_{k;t'} = \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot \mathbf{g}_{k;t'}} + \mathbf{y}^{-\mathbf{c}_{k;t}} \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot (\mathbf{g}_{k;t'} + B_o(-\mathbf{c}_{k;t}))}.$$

We can combine these two cases as follows:

$$F_{k;t'} = \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot \mathbf{g}_{k;t'}} (1 + \mathbf{y}^{|\mathbf{c}_{k;t}|} \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot B_o |\mathbf{c}_{k;t}|}).$$

Lastly, note that by definition, since $t' = \mu_k t$, $\mathbf{c}_{\ell+1} = \mathbf{c}_{k;t'} = -\mathbf{c}_{k;t}$, and so $|\mathbf{c}_{\ell+1}| = |\mathbf{c}_{k;t'}| = |\mathbf{c}_{k;t}|$. Hence

$$F_{k;t'} = \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot \mathbf{g}_{k;t'}} \left(1 + z_{\ell+1} \prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot B_o |\mathbf{c}_{\ell+1}|} \right) = \left(\prod_{j=1}^{\ell} L_j^{\tilde{\mathbf{c}}_j \cdot \mathbf{g}_{k;t'}} \right) L_{\ell+1}.$$

By Theorem 2.3.7, we know that $c'_\ell \cdot g_{k;t'} = 1$, so we have

$$F_{k;t'} = \prod_{j=1}^{\ell+1} L_j^{\tilde{\mathbf{c}}_j \cdot \mathbf{g}_{k;t'}}$$

as desired. □

Remark 3.1.2. When the initial exchange matrix B_o is skew-symmetric, $\tilde{\mathbf{c}}_j = \mathbf{c}_j$, and the theorem reduces to its form in Musiker (2019).

Example 3.1.3. Let $\mu = \mu_1 \mu_2 \mu_1$ and let

$$B_o = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.$$

We will compute the F-polynomial $F_{1,t_3}(y_1, y_2)$. The $\mathbf{c}, \mathbf{g}, \tilde{\mathbf{c}}$-vectors involved are as follows, which we obtained using Sage:

$$\mathbf{c}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \mathbf{c}_3 = \begin{bmatrix} -3 \\ -4 \end{bmatrix},$$
$$\tilde{\mathbf{c}}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \tilde{\mathbf{c}}_2 = \begin{bmatrix} -4 \\ -1 \end{bmatrix}, \tilde{\mathbf{c}}_3 = \begin{bmatrix} -3 \\ -1 \end{bmatrix},$$
$$\mathbf{g}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \mathbf{g}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \mathbf{g}_3 = \begin{bmatrix} -3 \\ 8 \end{bmatrix}.$$

The relevant dot products are

$$\tilde{\mathbf{c}}_1 \cdot \mathbf{g}_3 = 3,\ \tilde{\mathbf{c}}_2 \cdot \mathbf{g}_3 = 4,\ \tilde{\mathbf{c}}_3 \cdot \mathbf{g}_3 = 1,$$
$$\tilde{\mathbf{c}}_1 \cdot B_o|\mathbf{c}_2| = -1,\ \tilde{\mathbf{c}}_1 \cdot B_o|\mathbf{c}_3| = -4,\ \tilde{\mathbf{c}}_2 \cdot B_o|\mathbf{c}_3| = -4.$$

Therefore,

$$\begin{aligned} L_1 &= 1 + z_1 = 1 + \mathbf{y}^{|\mathbf{c}_1|} = 1 + y_1, \\ L_2 &= 1 + z_2 L_1^{\tilde{\mathbf{c}}_1 \cdot B_o|\mathbf{c}_2|} = 1 + \mathbf{y}^{|\mathbf{c}_2|}(1 + y_1)^{\tilde{\mathbf{c}}_1 \cdot B_o|\mathbf{c}_2|} = 1 + y_1 y_2 (1 + y_1)^{-1}, \\ L_3 &= 1 + z_3 L_1^{\tilde{\mathbf{c}}_1 \cdot B_o|\mathbf{c}_3|} L_2^{\tilde{\mathbf{c}}_2 \cdot B_o|\mathbf{c}_3|} = 1 + y_1^3 y_2^4 (1 + y_1)^{-4} (1 + y_1 y_2 (1 + y_1)^{-1})^{-4} \\ &= 1 + y_1^3 y_2^4 (1 + y_1 + y_1 y_2)^{-4}. \end{aligned}$$

Applying Gupta's formula then gives us

$$\begin{aligned}
F_{1,t_3}(y_1, y_2) &= L_1^{\tilde{\mathbf{c}}_1\cdot\mathbf{g}_3} L_2^{\tilde{\mathbf{c}}_2\cdot\mathbf{g}_3} L_3^{\tilde{\mathbf{c}}_3\cdot\mathbf{g}_3} \\
&= L_1^3 L_2^4 L_3 \\
&= (1+y_1)^3(1+y_1y_2(1+y_1)^{-1})^4(1+y_1^3y_2^4(1+y_1+y_1y_2)^{-4}) \\
&= \frac{(1+y_1+y_1y_2)^4+y_1^3y_2^4}{1+y_1} \\
&= 1+3y_1+3y_1^2+y_1^3+4y_1y_2+8y_1^2y_2+4y_1^3y_2+6y_1^2y_2^2+6y_1^3y_2^2+4y_1^3y_2^3+y_1^3y_2^4.
\end{aligned}$$

3.2 Connections between Gupta's Formula and Other Work

After Gupta posted her work as a preprint on arXiv, several experts contacted her and Professor Musiker to point out the equivalence of her formula above to two other known formulas for F-polynomials.

One of these correspondences brought Theorem 6.4 of the survey Keller (2012) to their attention, which was first proven in Nagao (2013). This theorem is about *quantum F-polynomials* in *quantum cluster algebras*, which are a non-commutative deformation of cluster algebras that recover a corresponding commutative cluster algebra when specialized appropriately. When specialized to the commutative setting, Nagao's result is Gupta's formula in disguise. An alternative derivation of Gupta's formula, in rather different presentation, is also suggested by Nakanishi (2021).

We will provide the details of the other connection that Gupta and Musiker were informed of, which is between Gupta's formula and path-ordered products. They are complete up to the proof of a root-theoretic result, which we hope to include in a future version of tis thesis.

To briefly recall, to use wall-crossings to compute the F-polynomial of a cluster variable with associated $\mathbf{g}$-vector λ, we may consider any path γ in the corresponding scattering diagram $\mathfrak{D}$ such that γ starts from the interior of a maximal cone that contains λ and ends in the interior of the dominant chamber. Then the F-polynomial is equal to $\mathbf{x}^{-\lambda}\mathfrak{p}_{\gamma,\mathfrak{D}}(\mathbf{x}^{\lambda})$.

Let us retain the notation $F_{i_\ell,t_\ell}, \mathbf{g}_\ell, \mathbf{c}_\ell, \tilde{\mathbf{c}}_\ell$ from the statement of Theorem 3.1.1. We shall need a series of lemmas that identifies $\mathbf{c}$, $\tilde{\mathbf{c}}$-vectors with quantities related to scattering diagrams. We thank Nathan Reading for his correspondence with us, which was helpful for formulating parts of the following lemma.

Lemma 3.2.1. 1. For all k,

$$\mathbf{c}_k = (\tilde{\mathbf{c}}_k)^\vee.$$

2. Let (δ_k, f_{δ_k}) be the wall associated to the cluster variable x_k. Then $f_{\delta_k} = 1 + \widehat{y}^{|\mathbf{c}_k|}$.

3. Let $t_k \in [0, -\infty)$ be such that $\gamma(t_k)$ is the unique point where γ crosses δ_k. Let

$$\epsilon_k = \begin{cases} 1 & \text{if } \langle \gamma'(t), |\mathbf{c}_k| \rangle < 0 \\ -1 & \text{if } \langle \gamma'(t), |\mathbf{c}_k| \rangle > 0 \end{cases}.$$

Then $\mathbf{c}_k = \varepsilon_k |\mathbf{c}_k|$.

4. $w(v_1, v_2) = v_1 \cdot B_o v_2$.

We believe these lemmas are true and we are still looking for the correct reference for some of these facts. We will assume these lemmas in the remainder of this section.

Recall that maximal cones of $\mathbf{g}\mathrm{Fan}(B)$ are in bijection with clusters of the cluster algebra defined by B. By going backwards along μ, we may associate μ with a path in $\mathfrak{D}$ from the maximal cone that corresponds to the final seed $t_\ell = \mu t_o$ to the dominant chamber which corresponds to the initial seed, such that:

1. this path crosses ℓ walls in total;

2. if we let the k-th wall crossed by γ be $(\mathfrak{d}_k, f_{\mathfrak{d}_k})$, then $f_{\mathfrak{d}_k} = 1 + \mathbf{y}^{|\mathbf{c}_{\ell+1-k}|} = 1 + z_{\ell+1-k}$ in Gupta's notation.

Given a path γ that crosses finitely many walls, let $C_\ell, C_{\ell-1}, \ldots, C_0$ denote the sequence of maximal cones that γ passes through (so that C_ℓ is the cone corresponding to the seed t_ℓ and C_0 is the dominant chamber). For $0 \le k \le \ell$, let γ_k denote a subpath of γ that starts from the interior of C_k and traces the rest of γ identically.

Since γ_0 crosses no walls, the automorphism $p_{\gamma_0, \mathfrak{D}}$ is the identity. Note also that $\gamma_\ell = \gamma$. Therefore, we have

$$F_{i_\ell, t_\ell}(\mathbf{y}) = \frac{p_{\gamma,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})}{\mathbf{x}^{\mathbf{g}_\ell}} = \frac{p_{\gamma_\ell,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})}{p_{\gamma_0,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})} = \prod_{k=1}^{\ell} \frac{p_{\gamma_k,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})}{p_{\gamma_{k-1},\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})}. \tag{3.1}$$

Proposition 3.2.2. Following the notation above,

$$\frac{p_{\gamma_k,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})}{p_{\gamma_{k-1},\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})} = L_k^{\tilde{\mathbf{c}}_k\cdot\mathbf{g}_\ell}.$$

Proof. We first prove that for $0 \le m \le k-1$,

$$p_{\gamma_m,\mathfrak{D}}(\mathbf{y}^{|\mathbf{c}_k|}) = \mathbf{y}^{|\mathbf{c}_k|}\prod_{j=1}^{m} L_j^{\tilde{\mathbf{c}}_j\cdot B_o|\mathbf{c}_k|}. \tag{3.2}$$

When $m = 0$, $p_{\gamma_m,\mathfrak{D}}$ is the identity and we interpret the empty product to be 1, so the claim follows. Now suppose that the claim is true for some $0 \le m < k-1$. By definition of wall-crossing and using Lemma 3.2.1,

$$p_{\gamma_{m+1},\mathfrak{D}}(\mathbf{y}^{|\mathbf{c}_k|}) = p_{\gamma_m,\mathfrak{D}}\left(\mathbf{y}^{|\mathbf{c}_k|}(1+z_{m+1})^{w(\epsilon_{m+1}|\mathbf{c}_{m+1}|^\vee,|\mathbf{c}_k|)}\right) = p_{\gamma_m,\mathfrak{D}}\left(\mathbf{y}^{|\mathbf{c}_k|}(1+z_{m+1})^{\tilde{\mathbf{c}}_{m+1}\cdot B_o|\mathbf{c}_k|}\right).$$

So

$$\begin{aligned}
p_{\gamma_{m+1},\mathfrak{D}}(\widehat{y}^{|\mathbf{c}_k|}) &= p_{\gamma_m,\mathfrak{D}}\left(\mathbf{y}^{|\mathbf{c}_k|}(1+z_{m+1})^{\tilde{\mathbf{c}}_{m+1}\cdot B_o|\mathbf{c}_k|}\right)\\
&= p_{\gamma_m,\mathfrak{D}}(\mathbf{y}^{|\mathbf{c}_k|})(1+p_{\gamma_m,\mathfrak{D}}(z_{m+1}))^{\tilde{\mathbf{c}}_{m+1}\cdot B_o|\mathbf{c}_k|}\\
&= \mathbf{y}^{|\mathbf{c}_k|}\prod_{j=1}^{m} L_j^{\tilde{\mathbf{c}}_j\cdot B_o|\mathbf{c}_k|}\cdot\left(1+\mathbf{y}^{|\mathbf{c}_{m+1}|}\prod_{j=1}^{m} L_j^{\tilde{\mathbf{c}}_j\cdot B_o|\mathbf{c}_{m+1}|}\right)^{\tilde{\mathbf{c}}_{m+1}\cdot B_o|\mathbf{c}_k|}\\
&= \mathbf{y}^{|\mathbf{c}_k|}\prod_{j=1}^{m+1} L_j^{\tilde{\mathbf{c}}_j\cdot B_o|\mathbf{c}_k|}.
\end{aligned}$$

This concludes the proof of Equation 3.2. For each $1 \le k \le \ell$, let ϕ_k denote the subpath of γ that starts at $\gamma_k(0)$ and ends at $\gamma_{k-1}(0)$. Since ϕ_k is isotopic to $\gamma_{k-1}^{-1}\circ\gamma_k$, we may reorganize the following ratio as follows:

$$\frac{p_{\gamma_k,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})}{p_{\gamma_{k-1},\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})} = \frac{p_{\gamma_{k-1},\mathfrak{D}}(p_{\gamma_{k-1}^{-1}\circ\gamma_k}(\mathbf{x}^{\mathbf{g}_\ell}))}{p_{\gamma_{k-1},\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})} = \frac{p_{\gamma_{k-1},\mathfrak{D}}(p_{\phi_k,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell}))}{p_{\gamma_{k-1},\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})} = p_{\gamma_{k-1},\mathfrak{D}}\left(\frac{p_{\phi_k,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})}{\mathbf{x}^{\mathbf{g}_\ell}}\right).$$

Notice that by the definition of wall-crossing automorphisms and using Lemma 3.2.1, since ϕ_k only crosses one wall whose decorating term is $1+z_k$, the argument $\frac{p_{\phi_k,\mathfrak{D}}(\mathbf{x}^{\mathbf{g}_\ell})}{\mathbf{x}^{\mathbf{g}_\ell}}$ can be rewritten as $(1+\mathbf{y}^{|c_k|})^{\langle\mathbf{g}_\ell,\epsilon_k|\mathbf{c}_k|^\vee\rangle} = (1+\mathbf{y}^{|c_k|})^{\tilde{\mathbf{c}}_k\cdot\mathbf{g}_\ell}$. Now using Equation 3.2, we have that

$$p_{\gamma_{k-1},\mathfrak{D}}((1+\mathbf{y}^{|c_k|})^{\tilde{\mathbf{c}}_k\cdot\mathbf{g}_\ell}) = \left(1+\mathbf{y}^{|\mathbf{c}_k|}\prod_{j=1}^{k-1} L_j^{\tilde{\mathbf{c}}_j\cdot B_o|\mathbf{c}_k|}\right)^{\tilde{\mathbf{c}}_k\cdot\mathbf{g}_\ell} = L_k^{\tilde{\mathbf{c}}_k\cdot\mathbf{g}_\ell}.$$

$\square$

The equality of $p_{\gamma,\mathfrak{D}}(\mathbf{y}^{\mathbf{g}_\ell})$ and F_{i_ℓ,t_ℓ} as given by Gupta's formula then follows from substituting into Equation 3.1 the identity given in Proposition 3.2.2.

Example 3.2.3. Let us again consider the example of the Kronecker quiver, which is skew-symmetric. The F-polynomial F_5, by Gupta's formula, is equal to

$$L_1^{\mathbf{c}_3\cdot\mathbf{g}_5} L_2^{\mathbf{c}_4\cdot\mathbf{g}_5} L_3^{\mathbf{c}_5\cdot\mathbf{g}_5} = L_1^3 L_2^2 L_3,$$

where

$$L_1 = 1 + y_1,$$

$$L_2 = 1 + y_1^2 y_2 L_1^{\mathbf{c}_3\cdot B_0|\mathbf{c}_4|} = 1 + y_1^2 y_2 (1+y_1)^{-2},$$

$$L_3 = 1 + y_1^3 y_2^2 L_1^{\mathbf{c}_3\cdot B_0|\mathbf{c}_5|} L_2^{\mathbf{c}_4\cdot B_0|\mathbf{c}_5|} = 1 + y_1^3 y_2^2 (1+y_1)^{-4}(1 + y_1^2 y_2 (1+y_1)^{-2})^{-2}.$$

Compare this with the path-ordered product. As suggested in the proof above, we compute

$$\frac{p_{\gamma_3,\mathfrak{D}}(x_1^{-3}x_2^4)}{x_1^{-3}x_2^4} = \prod_{k=1}^{3} \frac{p_{\gamma_k,\mathfrak{D}}(x_1^{-3}x_2^4)}{p_{\gamma_{k-1},\mathfrak{D}}(x_1^{-3}x_2^4)}.$$

Indeed,

$$\frac{p_{\gamma_1,\mathfrak{D}}(x_1^{-3}x_2^4)}{p_{\gamma_0,\mathfrak{D}}(x_1^{-3}x_2^4)} = (1+y_1)^{\mathbf{g}_5\cdot\mathbf{c}_3} = L_1^3;$$

$$\frac{p_{\gamma_2,\mathfrak{D}}(x_1^{-3}x_2^4)}{p_{\gamma_1,\mathfrak{D}}(x_1^{-3}x_2^4)} = p_{\gamma_1,\mathfrak{D}}((1+y_1^2y_2)^2) = (1 + y_1^2 y_2(1+y_1)^{-2})^2 = L_2^2;$$

$$\begin{aligned}
\frac{p_{\gamma_3,\mathfrak{D}}(x_1^{-3}x_2^4)}{p_{\gamma_2,\mathfrak{D}}(x_1^{-3}x_2^4)} &= p_{\gamma_2,\mathfrak{D}}(1 + y_1^3 y_2^2) \\
&= p_{\gamma_1,\mathfrak{D}}(1 + y_1^3 y_2^2 (1 + y_1^2 y_2)^{w(\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right], -\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right])}) \\
&= p_{\gamma_1,\mathfrak{D}}(1 + y_1^3 y_2^2 (1 + y_1^2 y_2)^2) \\
&= 1 + y_1^3 y_2^2 (1+y_1)^{w(\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right])}(1 + y_1^2 y_2 (1+y_1)^{-2})^2 \\
&= L_3.
\end{aligned}$$

3.3 Alternative Form of Gupta's Formula

One may wish to expand the product formula given in Theorem 3.1.1 into a multivariable power series. To do so, we first prove a lemma.

Lemma 3.3.1 (Exercise 3.2, Musiker (2019))**.** Given integers $h_1, \ldots, h_\ell$ and the same setup as Theorem 3.1.1,

$$\prod_{j=1}^{\ell} L_j^{h_j}\Big|_{z_i=\mathbf{y}^{|\mathbf{c}_i|}} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{h_j + \tilde{\mathbf{c}}_j \cdot \sum_{k=j+1}^{\ell} m_k B_o|\mathbf{c}_k|}{m_j} \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c}_j|}.$$

Proof. We prove the following claim by induction: for all $1 \leq i \leq \ell$,

$$\prod_{j=i}^{\ell} L_j^{h_j}\Big|_{z_i=\mathbf{y}^{|\mathbf{c}_i|}} = \sum_{(m_i,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{i-1} L_j^{\tilde{\mathbf{c}}_j \cdot \sum_{k=i}^{\ell} m_k B_o|\mathbf{c}_k|} \prod_{j=i}^{\ell} \binom{h_j + \tilde{\mathbf{c}}_j \cdot \sum_{k=j+1}^{\ell} m_k B_o|\mathbf{c}_k|}{m_j} \mathbf{y}^{\sum_{j=i}^{\ell} m_j|\mathbf{c}_j|}.$$

When $i = 1$, this claim specializes to our theorem.

First note that by the General Binomial Theorem, for any $h \in \mathbb{Z}$, we can write

$$\begin{aligned} L_k^h\Big|_{z_i=\mathbf{y}^{|\mathbf{c}_i|}} &= \left(1 + z_k \prod_{j=1}^{k-1} L_j^{\tilde{\mathbf{c}}_j \cdot B_o|\mathbf{c}_j|}\right)^h \Bigg|_{z_i=\mathbf{y}^{|\mathbf{c}_i|}} \\ &= \sum_{m\in\mathbb{Z}_{\geq 0}} \binom{h}{m} z_k^m \prod_{j=1}^{k-1} L_j^{\tilde{\mathbf{c}}_j \cdot mB_o|\mathbf{c}_j|}\Bigg|_{z_i=\mathbf{y}^{|\mathbf{c}_i|}} \\ &= \sum_{m\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{k-1} L_j^{\tilde{\mathbf{c}}_j \cdot mB_o|\mathbf{c}_k|} \binom{h}{m} \mathbf{y}^{m|\mathbf{c}_k|}. \end{aligned}$$

If we let $k = \ell$ and $h = h_\ell$, the above is precisely the base case $i = \ell$ of our claim.

Now suppose that our claim is true for $i+1$. Specializing our computation above to $k = i$ and $h = h_i + \tilde{\mathbf{c}}_i \cdot \sum_{k=i+1}^{\ell} m_k B_o|\mathbf{c}_k|$ gives us that

$$L_i^{h_i + \tilde{\mathbf{c}}_i \cdot \sum_{k=i+1}^{\ell} m_k B_o|\mathbf{c}_k|} = \sum_{m_i\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{i-1} L_j^{\tilde{\mathbf{c}}_j \cdot m_i B_o|\mathbf{c}_i|} \binom{h_i + \tilde{\mathbf{c}}_i \cdot \sum_{k=i+1}^{\ell} m_k B_o|\mathbf{c}_k|}{m_i} \mathbf{y}^{m_i|\mathbf{c}_k|}.$$

Using this, we compute that

$$\begin{aligned}
&\prod_{j=i}^{\ell} L_j^{h_j}\Big|_{z_i=\mathbf{y}^{|\mathbf{c}_i|}} \\
&= L_i^{h_i} \prod_{j=i+1}^{\ell} L_j^{h_j}\Big|_{z_i=\mathbf{y}^{|\mathbf{c}_i|}} \\
&= L_i^{h_i} \sum_{(m_{i+1},\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{i} L_j^{\tilde{\mathbf{c}}_j\cdot\sum_{k=i+1}^{\ell} m_k B_o|\mathbf{c}_k|} \prod_{j=i+1}^{\ell} \binom{h_j + \tilde{\mathbf{c}}_j \cdot \sum_{k=j+1}^{\ell} m_k B_o|\mathbf{c}_k|}{m_j} \mathbf{y}^{\sum_{j=i+1}^{\ell} m_j|\mathbf{c}_j|} \\
&= \sum_{(m_{i+1},\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{i-1} L_j^{\tilde{\mathbf{c}}_j\cdot\sum_{k=i+1}^{\ell} m_k B_o|\mathbf{c}_k|} L_i^{h_i+\tilde{\mathbf{c}}_i\cdot\sum_{k=i+1}^{\ell} m_k B_o|\mathbf{c}_k|} \prod_{j=i+1}^{\ell} \binom{h_j + \tilde{\mathbf{c}}_j \cdot \sum_{k=j+1}^{\ell} m_k B_o|\mathbf{c}_k|}{m_j} \mathbf{y}^{\sum_{j=i+1}^{\ell} m_j|\mathbf{c}_j|} \\
&= \sum_{(m_i,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{i-1} L_j^{\tilde{\mathbf{c}}_j\cdot\sum_{k=i}^{\ell} m_k B_o|\mathbf{c}_k|} \prod_{j=i}^{\ell} \binom{h_j + \tilde{\mathbf{c}}_j \cdot \sum_{k=j+1}^{\ell} m_k B_o|\mathbf{c}_k|}{m_j} \mathbf{y}^{\sum_{j=i}^{\ell} m_j|\mathbf{c}_j|}.
\end{aligned}$$

This completes our proof for the inductive step. □

Lemma 3.3.1 and Theorem 3.1.1 together imply the following alternative form of Gupta's formula.

Theorem 3.3.2. Under the same conditions as Theorem 3.1.1,

$$F_{i_\ell;t_\ell}(\mathbf{y}) = \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\tilde{\mathbf{c}}_j \cdot (\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_o|\mathbf{c}_k|)}{m_j} \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c}_j|}.$$

We refer the reader to Section 4.2 for examples of this theorem.

Chapter 4

Rank-Two Cluster Algebras

4.1 Rank-Two Basics

Rank-two is much simpler than the general case because there are only two mutation directions at any cluster. We specialize the general definitions given in Chapter 2 here to give a clearer picture of rank-two cluster algebras.

A 2-by-2 matrix $B = (b_{ij})$ is skew-symmetrizable if there exist $\delta_1, \delta_2 > 0$ such that $\delta_1 b_{12} = -\delta_2 b_{21}$, $\delta_1 b_{11} = -\delta_1 b_{11}$, and $\delta_2 b_{22} = -\delta_2 b_{22}$. So a nonzero skew-symmetrizable two-by-two matrix must be of the form

$$B = \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix}, \text{ where } b_{12}b_{21} < 0.$$

Thus, up to relabeling, rank-two cluster algebras are defined by an initial exchange matrix of the form

$$B_o = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix},$$

where $b, c > 0$. Given $b, c > 0$, the corresponding rank-two cluster algebra is denoted by $\mathcal{A}(b, c)$. As mentioned in the introduction, in this notation, the r-Kronecker cluster algebra is $\mathcal{A}(r, r)$, and the Kronecker cluster algebra is $\mathcal{A}(2, 2)$.

By the classification of rank-two root systems, we know that a rank-two cluster algebra is of finite type, affine type and indefinite type if and only if $bc \leq 3$, $bc = 4$, and $bc > 4$ respectively. When $bc \geq 4$, let $\mu_+, \mu_+(m), \mu_-, \mu_-(m)$ and the related numbering of cluster variables and $\mathbf{d}, \mathbf{c}, \mathbf{g}$-vectors be defined similarly as in Examples 2.1.6 and 2.3.6. Our definitions in the Kronecker case apply verbatim here because when $bc \geq 4$, μ_+

and μ_- will produce distinct seeds at each step, and the exchange graph for $\mathcal{A}(b,c)$ is exactly $\mathbb{T}_2$, the infinite line graph with countably many vertices, just like the Kronecker exchange graph (Figure 2.5).

We will now summarize the mutation rule for the rank-two cluster variables and provide formulas for their $\mathbf{d}, \mathbf{c}, \mathbf{g}$-vectors. Note that our calculations specialize to Example 2.1.5 when $b = c = 1$, and to Example 2.1.6 when $b = c = 2$.

For $m \in \mathbb{Z}$, let $C_{m,b,c}$ be the sequence defined by $C_{-1,b,c} = -1$, $C_{0,b,c} = 0$, and

$$C_{m,b,c} = \begin{cases} cC_{m-1,b,c} - C_{m-2,b,c}, & \text{if } m \text{ is even;} \\ bC_{m-1,b,c} - C_{m-2,b,c}, & \text{if } m \text{ is odd.} \end{cases}$$

Let $A_m = C_{m,b,c}$ and $B_m = C_{m,c,b}$. Note that by design, both A_m and B_m are positive for $m \geq 1$.

We can calculate that

$$\mathbf{c}_{-1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ \mathbf{c}_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \ \mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and for $k \geq 0$,

$$\mathbf{c}_{k+2} = \begin{bmatrix} -B_k \\ -A_{k-1} \end{bmatrix}, \mathbf{c}_{-k-1} = \begin{bmatrix} B_{k-1} \\ A_k \end{bmatrix}.$$

Knowing the relevant $\mathbf{c}$-vectors, we deduce that the cluster variables satisfy the following explicit recurrence:

$$\begin{aligned} x_0 x_2 &= x_1^b + y_2, \\ x_{-1} x_1 &= x_0^c y_1 + 1, \end{aligned}$$

and for $k \geq 0$,

$$x_{k+3} x_{k+1} = \begin{cases} x_{k+2}^b + y_1^{B_{k+1}} y_2^{A_k} & \text{if } k \text{ is odd,} \\ x_{k+2}^c + y_1^{B_{k+1}} y_2^{A_k} & \text{if } k \text{ is even;} \end{cases} \tag{4.1}$$

$$x_{-k-2} x_{-k} = \begin{cases} x_{-k-1}^b + y_1^{B_k} y_2^{A_{k+1}} & \text{if } k \text{ is even,} \\ x_{-k-1}^c + y_1^{B_k} y_2^{A_{k+1}} & \text{if } k \text{ is odd.} \end{cases}$$

Often, we specialize so that $y_i = 1$. The cluster variables after the specialization satisfy a nicer recurrence: for all $m \in \mathbb{Z}$,

$$x_{m+1} x_{m-1} = \begin{cases} x_m^b + 1 & \text{if } m \text{ is odd,} \\ x_m^c + 1 & \text{if } m \text{ is even.} \end{cases}$$

Using this recurrence, we have that

$$\mathbf{g}_{-1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ \mathbf{g}_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \ \mathbf{g}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{g}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and for $k \geq 0$ and $k \leq -2$,

$$\mathbf{g}_{k+2} = \begin{cases} b\mathbf{g}_{k+1} - \mathbf{g}_k & \text{if } k \text{ is even,} \\ c\mathbf{g}_{k+1} - \mathbf{g}_k & \text{if } k \text{ is odd.} \end{cases}$$

Hence, for $k \geq 0$,

$$\mathbf{g}_{k+1} = \begin{bmatrix} -B_{k-1} \\ A_k \end{bmatrix}, \ \mathbf{g}_{-k} = \begin{bmatrix} -B_k \\ A_{k-1} \end{bmatrix}.$$

Lastly, we consider **d**-vectors. Note that in general, for cluster algebra with rank n, for $1 \leq i \leq n$, we conventionally understand $\mathbf{d}_i = -\mathbf{e}_i$. In the rank-two case, we have that for $k \geq 0$,

$$\mathbf{d}_{k+2} = \begin{bmatrix} B_k \\ A_{k-1} \end{bmatrix}, \ \mathbf{d}_{-k+1} = \begin{bmatrix} B_{k-1} \\ A_k \end{bmatrix}.$$

4.2 The r-Kronecker Cluster Algebra and Gupta's Formula

When $b = c = r$, let $c_m = C_{m,r,r}$. To recall, this is the sequence defined by $c_{-1} = -1$, $c_0 = 0$, and $c_m = rc_{m-1} - c_{m-2}$. Note that $c_1 = 1$ for all r. We shall need this sequence frequently for some of the combinatorial models that are geared towards the skew-symmetric case. We prove some more properties of this sequence in the Appendix (Section A).

For $\ell \in \mathbb{Z}$ and $M, N \geq 0$, let $C_{M,N}^{(\ell,b,c)}$ denote the coefficient of $y_1^M y_2^N$ in the F-polynomial $F_\ell(y_1, y_2)$ of $\mathcal{A}(b, c)$. In other words,

$$F_\ell(y_1, y_2) = \sum_{M,N \geq 0} C_{M,N}^{(\ell,b,c)} y_1^M y_2^N.$$

When $b = c = r$, we follow the notation in our REU report and write $C_{M,N}^{(\ell,r)} = C_{M,N}^{(\ell,b,c)}$.

With the computation of $\mathbf{c}, \mathbf{g}$-vectors from Section 4.1, it is straightforward to apply Gupta's formula to the r-Kronecker $\mathcal{A}(r, r)$. This was done in Gupta (2018).

Theorem 4.2.1 (Theorem 3.1, Gupta (2018))**.** For $1 \leq k \leq \ell$, let L_k be the rational function in y_1, y_2 as defined in Theorem 3.1.1 for the mutation sequence $\mu_+(\ell)$. Then for $\ell > 0$, the F-polynomial $F_{\ell+2}$ has the following formula:

$$F_{\ell+2}(y_1, y_2) = L_1^{c_\ell} L_2^{c_{\ell-1}} \cdots L_\ell \tag{4.2}$$

$$= \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}^\ell} \prod_{i=1}^{\ell} \binom{c_{\ell-i+1} - r\sum_{j=i+1}^{\ell} c_{j-i}m_j}{m_i} y_1^M y_2^N \tag{4.3}$$

where

$$M = c_1m_1 + c_2m_2 + \cdots + c_\ell m_\ell,$$
$$N = c_1m_2 + c_2m_3 + \cdots + c_{\ell-1}m_\ell.$$

In other words,

$$C_{M,N}^{(\ell,r)} := \sum_{\substack{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}^\ell \\ c_1m_1+c_2m_2+\cdots+c_\ell m_\ell=M \\ c_1m_2+c_2m_3+\cdots+c_{\ell-1}m_\ell=N}} \prod_{i=1}^{\ell} \binom{c_{\ell-i+1} - r\sum_{j=i+1}^{\ell} c_{j-i}m_j}{m_i}.$$

When $r = 2$, the sequence c_n is particularly simple; in fact, $c_n = n$. Thus, (positively-indexed) F-polynomials of the Kronecker quiver have the following formula:

$$F_{\ell+2}(y_1, y_2) = L_1^{\ell} L_2^{\ell-1} \cdots L_\ell \tag{4.4}$$

$$= \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}^\ell} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - \sum_{j=i+1}^{\ell} 2(j-i)m_j}{m_i} y_1^M y_2^N. \tag{4.5}$$

Based on experimental data, we observed the following phenomenon regarding $C_{M,N}^{(\ell,r)}$. We were hopeful that the resolution of this conjecture might bring new insight to whether a simple formula for the $C_{M,N}^{(\ell,r)}$'s is possible when $r > 2$.

Conjecture 1. If $c_{\ell+1} - rM < 0$, then

$$\sum_N (-1)^N C_{M,N}^{(\ell,r)} = \sum_{N=0}^{\left\lfloor M\frac{c_{\ell-1}}{c_\ell}\right\rfloor} (-1)^N C_{M,N}^{(\ell,r)} = 0.$$

Example 4.2.2. If we substitute the binomial coefficient formula for $C_{M,N}^{(\ell,2)}$, we can verify this conjecture for $r = 2$. Now consider the case where $\ell = 4$, $r = 3$. Then $c_{\ell+1} = c_5 = 54$, and $c_{\ell+1} - rM = 55 - 3M$. So it suffices to have $M \geq 19$. Based on data, the only non-zero coefficients $C_{M,N}^{(4,3)}$ are as given in Table 4.1. One may check that the alternating sum of each of these rows is indeed zero.

M N	0	1	2	3	4	5	6	7	8
19	210	1224	2940	3732	2655	1020	177	6	
20	21	144	420	675	645	366	114	15	
21	1	8	28	56	70	56	28	8	1

Table 4.1 Data for $C_{M,N}^{(4,3)}$ when $55 - 3M < 0$.

In addition to the alternating sum phenomenon, in our investigation of the tuples that are summed over to calculate $C_{M,N}^{(\ell,r)}$, there were some mysterious groups of tuples whose total contribution were particularly nice.

Example 4.2.3. We record here all the examples that we observed. Let

$$c(m_1, m_2, m_3, m_4, m_5) = \binom{55-3(m_2+3m_3+8m_4+21m_5)}{m_1}\binom{21-3(m_3+3m_4+8m_5)}{m_2}\binom{8-3(m_4+3m_5)}{m_3}\binom{3-3m_5}{m_4}\binom{1}{m_5}.$$

Then

$$C_{M,N}^{(5,3)} := \sum_{\substack{(m_1,\ldots,m_5)\in\mathbb{Z}_{\geq 0}^5 \\ m_1+3m_2+8m_3+21m_4+55m_5=M \\ m_2+3m_3+8m_4+21m_5=N}} c(m_1, m_2, m_3, m_4, m_5).$$

Consider the following sets of tuples:

$$\begin{aligned}
S_1 &= \{(0,20,0,0,0),(1,17,1,0,0),(2,14,2,0,0),(3,11,3,0,0),\\
&\qquad (4,8,4,0,0),(5,5,5,0,0),(6,2,6,0,0)\};\\
S_2 &= \{(3,12,0,1,0),(4,9,1,1,0),(5,6,2,1,0),(6,3,3,1,0),(7,0,4,1,0)\};\\
S_3 &= \{(0,17,1,0,0),(1,14,2,0,0),(2,11,3,0,0),\\
&\qquad (3,8,4,0,0),(4,5,5,0,0),(5,2,6,0,0)\}.
\end{aligned}$$

Both S_1 and S_2 consist of tuples that contribute to $C_{60,20}^{(5,3)}$, and S_3 consists of tuples that contribute to $C^{(5,3)}_{59,20}$. Then curiously,

$$\sum_{T\in S_1} c(T) = \binom{\binom{11}{2}}{2} = 1485,$$

$$\sum_{T\in S_2} c(T) = -\binom{\binom{11}{2}}{2} = -1485,$$

$$\sum_{T\in S_3} c(T) = \binom{\binom{10}{2}}{2} = 990.$$

The reader might notice that we were investigating M, N's for which $C^{(\ell,r)}_{M,N} = 0$; indeed $C^{(5,3)}_{60,20} = C^{(5,3)}_{59,20} = 0$. This is motivated by the same line of investigation in Lin, Feiyang (2020), where we hope to explicitly demonstrate that only finitely many $C^{(\ell,r)}_{M,N}$'s are nonzero for a fixed pair of ℓ and r, which demonstrates that Gupta's formula indeed produces polynomials. Observing these examples, we tried to find a general pattern where the contribution of a family of tuples would sum up to $\binom{\binom{m}{2}}{2}$ for some m, but failed. We also tried to use hypergeometric series techniques to prove that these specific sums evaluate in this manner, but were unable to evaluate the corresponding hypergeometric series using general theorems.

Lastly, given the above calculation for $\mathcal{A}(r, r)$, in our future research, we would also like to apply Gupta's formula to the more general skew-symmetrizable rank-two cases, i.e. to the cluster algebras denoted as $\mathcal{A}(b, c)$.

Question 4.2.4. How does Gupta's formula specialize for F-polynomials of $\mathcal{A}(b, c)$ where $b \neq c$? In particular, what is Gupta's formula for $\mathcal{A}(1, 4)$? Are the numbers $\tilde{\mathbf{c}}_k \cdot \mathbf{g}_\ell$ and $\tilde{\mathbf{c}}_k \cdot B_0|\mathbf{c}_\ell|$ somehow similarly tame as in the Kronecker case? What are the implications for Remark 5.3.1?

4.3 Support of F-Polynomials of $\mathcal{A}(r, r)$

Given a polynomial

$$F(y_1, \ldots, y_n) = \sum_{(e_1,\ldots,e_n)\in\mathbb{Z}_{\geq 0}} c_{e_1,\ldots,e_n} y_1^{e_1} \cdots y_n^{e_n},$$

its *support* is the set $\{(e_1, \ldots, e_n) \in \mathbb{Z}_{\geq 0} : c_{e_1,\ldots,e_n} \neq 0\}$. In Lin, Feiyang (2020), we conjectured that the support of F-polynomials of the r-Kronecker cluster algebra is exactly the integer points within a certain triangle.

Conjecture 2 (Conjecture 1 and 2, Lin, Feiyang (2020)). $C^{(\ell,r)}_{M,N} > 0$ if and only if $0 \le M \le c_\ell$ and $0 \le N \le \frac{c_{\ell-1}}{c_\ell}M$.

We may specialize the notation of Lee et al. (2014) for greedy elements to cluster variables of $\mathcal{A}(r,r)$ as follows: for $\ell \ge 1$,

$$x[c_\ell, c_{\ell-1}] = x_1^{-c_\ell} x_2^{-c_{\ell-1}} \sum_{p,q\ge 0} c(p,q) x_1^{rp} x_2^{rq}.$$

Since

$$F_\ell(y_1,y_2) = \sum_{M,N\ge 0} C^{(\ell,r)}_{M,N} y_1^M y_2^N,$$

by the explicit formula for F-polynomial and cluster variable coefficients for $\mathcal{A}(r,r)$ given in Lee and Schiffler (2013), we have

$$c(N, c_\ell - M) = C^{(\ell,r)}_{M,N}.$$

Therefore, Conjecture 2 says that the *pointed support* of $x[c_\ell, c_{\ell-1}]$, namely the set $\{(p,q) : c(p,q) \ne 0\}$, is the set of integer points within the triangular region with vertices $(0,0), (0,c_\ell), (c_{\ell-1}, 0)$.

On the other hand, applied to $x[c_\ell, c_{\ell-1}]$, case (6) of Proposition 4.1 in Lee et al. (2014) says that the pointed support of $x[c_\ell, c_{\ell-1}]$ is contained in the quadrilateral with vertices

$$(0,0), (0,c_\ell), (\frac{c_\ell}{r}, \frac{c_{\ell-1}}{r}), (c_{\ell-1}, 0).$$

As illustrated in the figure below, the quadrilateral region suggested by Lee et al. (2014) is always slightly larger than the triangle given by our conjecture, but they appear to include the same set of lattice points.

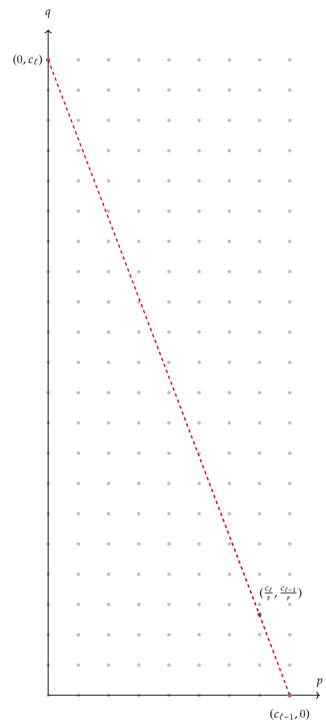


Figure 4.1 Comparison of two different regions that contain the support: the triangle whose vertices are $(0,0),(0,21),(8,0)$ and the quadrilateral whose vertices are $(0,0),(0,21),(\frac{21}{3},\frac{8}{3}),(8,0)$, which almost cannot be distinguished on this figure

The following theorem says that Proposition 4.1 of Lee et al. (2014) and our inequalities delineate the same set of integral points.

Theorem 4.3.1. Let $p, q \in \mathbb{Z}_{\geq 0}$. Then when $0 \leq p \leq \frac{c_\ell}{r}$, we have

$$p \leq \frac{c_{\ell-1}}{c_\ell}(c_\ell - q) \Leftrightarrow (p,q) \text{ lies below the segment } [(0,c_\ell),(c_\ell/r, c_{\ell-1}/r)];$$

and when $\frac{c_\ell}{r} \leq p \leq c_{\ell-1}$,

$$p \leq \frac{c_{\ell-1}}{c_\ell}(c_\ell - q) \Leftrightarrow (p,q) \text{ lies below the segment } [(c_\ell/r, c_{\ell-1}/r),(c_{\ell-1},0)].$$

Proof. We prove the first claim first. Let $s = \frac{p}{c_\ell c_{\ell-1}}$. First note that when $0 \leq p \leq \frac{c_\ell}{r}$, $s = \frac{p}{c_\ell c_{\ell-1}} \leq \frac{1}{rc_{\ell-1}} < \frac{1}{c_\ell}$. We will use this fact later. For (p,q) to lie below the segment $[(0,c_\ell),(c_\ell/r,c_{\ell-1}/r)]$, we need

$$\frac{q-c_\ell}{p} \le \frac{c_{\ell-1}/r - c_\ell}{c_\ell/r},$$

or equivalently, $c_\ell - q \ge p\frac{c_{\ell+1}}{c_\ell}$. Therefore, to show the first equivalence, it suffices to show that when $0 \le p \le \frac{c_\ell}{r}$, $\left\lceil p\frac{c_{\ell+1}}{c_\ell}\right\rceil = \left\lceil p\frac{c_\ell}{c_{\ell-1}}\right\rceil$. Using Proposition A.0.3, we have

$$\frac{c_{\ell+1}}{c_\ell} = \frac{c_{\ell+1}c_{\ell-1}}{c_\ell c_{\ell-1}} = \frac{c_\ell^2-1}{c_\ell c_{\ell-1}} = \frac{c_\ell}{c_{\ell-1}} - \frac{1}{c_\ell c_{\ell-1}},$$

therefore $p\frac{c_\ell}{c_{\ell-1}} = p\frac{c_{\ell+1}}{c_\ell} + s$ and it suffices to show that $\left\lceil p\frac{c_{\ell+1}}{c_\ell}+s\right\rceil = \left\lceil p\frac{c_{\ell+1}}{c_\ell}\right\rceil$.

But note that $p\frac{c_{\ell+1}}{c_\ell}$ cannot be an integer: by Proposition A.0.4, $p\frac{c_{\ell+1}}{c_\ell} \in \mathbb{Z}$ only if $c_\ell \mid p$, but $p \le c_\ell/r$. So $\left\lceil p\frac{c_{\ell+1}}{c_\ell}\right\rceil - p\frac{c_{\ell+1}}{c_\ell} \ge 1/c_\ell$. Since $s < \frac{1}{c_\ell}$, we have $\left\lceil p\frac{c_{\ell+1}}{c_\ell}+s\right\rceil = \left\lceil p\frac{c_{\ell+1}}{c_\ell}\right\rceil$ as desired.

The proof of the second claim is similar. Let $t = \frac{c_{\ell-1}-p}{c_{\ell-1}c_{\ell-2}}$. Note that when $\frac{c_\ell}{r} \le p \le c_{\ell-1}$, $c_{\ell-1}-p \le c_{\ell-1} - \frac{c_\ell}{r} = \frac{c_{\ell-2}}{r}$, and so $t = \frac{c_{\ell-1}-p}{c_{\ell-1}c_{\ell-2}} \le \frac{1}{rc_{\ell-1}} < \frac{1}{c_{\ell-1}}$. We will use this fact later. For (p,q) to lie below the segment $[(c_\ell/r,c_{\ell-1}/r),(c_{\ell-1},0)]$, we need

$$\frac{0-q}{c_{\ell-1}-p} \ge \frac{0-\frac{c_{\ell-1}}{r}}{c_{\ell-1}-\frac{c_\ell}{r}},$$

or equivalently, $q \le (c_{\ell-1}-p)\frac{c_{\ell-1}}{c_{\ell-2}}$. By inspecting the graph of $p = \frac{c_{\ell-1}}{c_\ell}(c_\ell - q)$, we find that

$$p \le \frac{c_{\ell-1}}{c_\ell}(c_\ell - q) \Leftrightarrow q \le (c_{\ell-1}-p)\frac{c_\ell}{c_{\ell-1}}.$$

Therefore, to show the equivalence, it suffices to show that when $\frac{c_\ell}{r} \le p \le c_{\ell-1}$,

$$\left\lfloor (c_{\ell-1}-p)\frac{c_{\ell-1}}{c_{\ell-2}}\right\rfloor = \left\lfloor (c_{\ell-1}-p)\frac{c_\ell}{c_{\ell-1}}\right\rfloor.$$

Recall that $\frac{c_\ell}{c_{\ell-1}} = \frac{c_{\ell-1}}{c_{\ell-2}} - \frac{1}{c_{\ell-1}c_{\ell-2}}$. Therefore, $(c_{\ell-1}-p)\frac{c_{\ell-1}}{c_{\ell-2}} = (c_{\ell-1}-p)\frac{c_\ell}{c_{\ell-1}} + t$ and it suffices to show that $\left\lfloor (c_{\ell-1}-p)\frac{c_\ell}{c_{\ell-1}} + t\right\rfloor = \left\lfloor (c_{\ell-1}-p)\frac{c_\ell}{c_{\ell-1}}\right\rfloor$. But by Proposition A.0.4, since $p > 0$ in this case and so $c_{\ell-1}-p < c_{\ell-1}$, the quantity $(c_{\ell-1}-p)c_\ell/c_{\ell-1}$ cannot be an integer. So we have $\lceil (c_{\ell-1}-p)c_\ell/c_{\ell-1}\rceil - (c_{\ell-1}-p)c_\ell/c_{\ell-1} \ge 1/c_{\ell-1}$. Since $t < \frac{1}{c_{\ell-1}}$, we have $\left\lfloor (c_{\ell-1}-p)\frac{c_\ell}{c_{\ell-1}} + t\right\rfloor = \left\lfloor (c_{\ell-1}-p)\frac{c_\ell}{c_{\ell-1}}\right\rfloor$ as desired. □

4.4 Preliminaries: Maximal Dyck Paths and Christoffel Words

For the remainder of this chapter, we discuss two formulas for cluster variables of $\mathcal{A}(r,r)$, due to Lee and Schiffler (2013) and Lee et al. (2014) respectively, and then introduce a conjectural bijection between two families of combinatorial objects which are involved, which implies the equivalence the two formulas. When speaking of the cluster variables $x_k \in \mathcal{A}(b,c)$, we always specialize to $y_1 = y_2 = 1$.

Both Lee and Schiffler (2013) and Lee et al. (2014) make use of the *maximal Dyck path.* Denote by $R_{m\times n}$ the rectangle with vertices $(0,0)$, $(m,0)$, $(0,n)$ and (m,n). A *Dyck path* in $R_{m\times n}$ is a lattice path from $(0,0)$ to (m,n) that only goes up and right and never goes above the diagonal line segment from $(0,0)$ to (m,n). In other words, if (x,y) is a point in a Dyck path of $R_{m\times n}$, then $\frac{y}{x} \le \frac{n}{m}$. Visually, a Dyck path is *maximal* if when viewed as starting from $(0,0)$, the path goes up whenever it can. In other words, if y is the largest vertical coordinate of points on $\mathcal{D}_{m\times n}$ with horizontal coordinate x, then $y = \lfloor x\frac{n}{m}\rfloor$. Denote by $\mathcal{D}_{m\times n}$ the unique maximal Dyck path in $R_{m\times n}$ and label its $m+n$ edges in order as $\alpha_1,\ldots,\alpha_{m+n}$. For $1 \le j \le n$, let μ_j denote the j-th vertical edge. For $1 \le k \le m$, let ρ_k denote the k-th horizontal edge. Let $\mathcal{U}_{m\times n} = \{\mu_1,\ldots,\mu_n\}$ be the set of vertical edges and let $\mathcal{R}_{m\times n} = \{\rho_1,\ldots,\rho_m\}$ be the set of horizontal edges. (The letters μ and ρ stand for up and right.) Let $\mathcal{E}_{m\times n} = \mathcal{U}_{m\times n} \sqcup \mathcal{R}_{m\times n} = \{\alpha_1,\alpha_2,\ldots,\alpha_{m+n}\}$. For $0 \le i \le m+n$, let ω_i denote the i-th vertex in the maximal Dyck path, so that $\omega_0 = (0,0)$, $\omega_{m+n} = (m,n)$. Let $v_0 = (0,0)$ and for $0 < j \le n$, let v_j denote the upper endpoint of μ_j.

The following notations follow that of Lee et al. (2014). Given two vertices E, F on a lattice path $\mathcal{D}$, let EF denote the subpath of $\mathcal{D}$ which begins with E and ends with F. Let $(EF)_1$, $(EF)_2$ denote the sets of horizontal and vertical edges in the subpath EF respectively. Let EF° denote the set of lattice points that lie strictly between E and F on the subpath EF; in this context, we identify ω_0 and ω_{m+n} to be the same vertex.

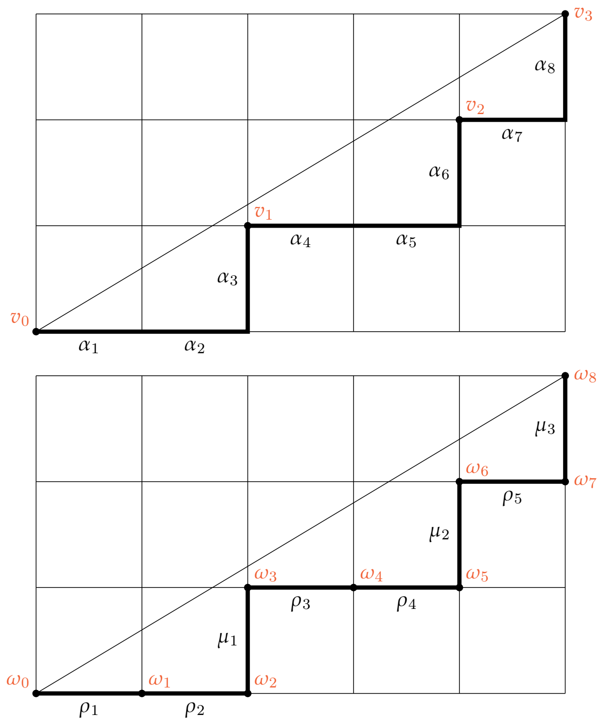


Figure 4.2 Illustration of our vertex and edge labeling conventions on $\mathcal{D}_{5\times 3}$. For clarity, vertices are labeled orange. The relevant edge sets are $\mathcal{U}_{5\times 3} = \{\mu_1, \mu_2, \mu_3\}$, $\mathcal{R}_{5\times 3} = \{\rho_1, \rho_2, \ldots, \rho_5\}$, and $\mathcal{E}_{5\times 3} = \mathcal{U}_{5,3} \sqcup \mathcal{R}_{5\times 3} = \{\alpha_1, \alpha_2, \ldots, \alpha_8\}$. We can check that the largest y-coordinate of points with x-coordinate 4 is $\lfloor 4 \cdot \frac{3}{5} \rfloor = 2$. Alternatively, there are 2 vertical edges in the first 6 steps because $\left\lfloor 6 \cdot \frac{3}{3+5} \right\rfloor = 2$. If we let $E = \omega_6$, $F = \omega_3$, then $EF^\circ = \{\omega_7, \omega_8, \omega_1, \omega_2\}$

We also need to borrow the language of Christoffel words.

Definition 4.4.1 (Christoffel Word)**.** Given positive integers a and b, the (lower) Christoffel word $w(a, b)$ is the word in the alphabet $\{x, y\}$ with a x's and b y's, such that the number of y's in the first k letters is equal to $\left\lfloor \frac{bk}{a+b} \right\rfloor$.

The word $w(a, b)$ compactly encodes $\mathcal{D}_{a\times b}$: starting at the lower corner of $\mathcal{R}_{a\times b}$, x's correspond to rightward steps and y's correspond to upward steps.

Example 4.4.2. The path $\mathcal{D}_{5\times 3}$, which is pictured in Figure 4.2.

Given two words α, β, we define $\alpha\beta = \alpha \cdot \beta$ to be their concatenation. Suppose a, b are coprime. Then by Lemma 1.3 of Berstel et al. (2008), there is a unique point $C = (i, j)$ on $\mathcal{D}_{a\times b}$ such that it attains the shortest vertical distance to the diagonal connecting $(0,0)$ and (a, b), namely $\frac{ib-ja}{a} = \frac{1}{a}$.

Definition 4.4.3. Let a, b be positive, coprime integers, and let C be the unique point on $\mathcal{D}_{a\times b}$ vertically closest to the diagonal. The *standard factorization* of a Christoffel word $w(a, b)$ is $w(a, b) = \alpha\beta$, where α encodes the portion of $\mathcal{D}_{a\times b}$ from $(0,0)$ to C, and β the portion from C to (a, b).

Theorem 4.4.4 (Proposition 3.2 and Theorem 3.3, Berstel et al. (2008))**.** For a, b coprime, the standard factorization of the Christoffel word $w(a, b)$ is the unique factorization of $w(a, b)$ into two Christoffel words.

Applying this theorem to Christoffel words of specific slopes, we obtain the following result about the structure of some maximal Dyck paths that we are interested in.

Lemma 4.4.5. For $n \geq 2$,

$$w(c_{n+1}, c_n) = w(c_n, c_{n-1})^{r-1} \cdot w(c_n - c_{n-1}, c_{n-1} - c_{n-2})$$

$$w(c_{n+1} - c_n, c_n - c_{n-1}) = w(c_n, c_{n-1})^{r-2} \cdot w(c_n - c_{n-1}, c_{n-1} - c_{n-2})$$

Proof. We prove the following statement instead: for $0 \leq k < r$, let $\alpha_k = w(c_{n+1} - kc_n, c_n - kc_{n-1})$. Then

$$\alpha_k = w(c_n, c_{n-1})^{r-k-1} \cdot w(c_n - c_{n-1}, c_{n-1} - c_{n-2}).$$

This claim specializes to our lemma when $k = 0$ and $k = 1$.

We proceed by induction. When $k = r - 1$, since

$$c_{n+1} - (r-1)c_n = (rc_n - c_{n-1}) - (r-1)c_n = c_n - c_{n-1}$$

and $r - k - 1 = 0$, the statement is trivially true. Suppose that the statement is true for some $1 \leq k < r$. We claim that $\alpha_{k-1} = w(c_n, c_{n-1}) \cdot \alpha_k$. Let $C = (i, j) = (c_n, c_{n-1})$ and let $a = c_{n+1} - (k-1)c_n$, $b = c_n - (k-1)c_{n-1}$. It suffices to check that $C = (i, j) = (c_n, c_{n-1})$ is the point that defines the standard factorization of $\alpha_{k-1} = w(a, b)$. Indeed,

$$\begin{aligned} ib - ja &= c_n(c_n - (k-1)c_{n-1}) - c_{n-1}(c_{n+1} - (k-1)c_n) \\ &= c_n^2 - c_{n-1}c_{n+1} \\ &= 1, \end{aligned}$$

where the last equality is due to Proposition A.0.3. Applying the inductive hypothesis, we obtain that

$$\alpha_{k-1} = w(c_n, c_{n-1}) \cdot \alpha_k = w(c_n, c_{n-1})^{r-k} \cdot w(c_n - c_{n-1}, c_{n-1} - c_{n-2})$$

as desired. □

We shall also need the following fact.

Proposition 4.4.6 (Proposition 4.2, Berstel et al. (2008))**.** Suppose $a, b \in \mathbb{Z}$ are coprime. Then $w(a,b) = xuy$ with u a palindrome.

Example 4.4.7. When $r = 3$, $c_0, c_1, c_2, c_3 = 0, 1, 3, 8$. So when $n = 2$ and $r = 3$, the first part of Lemma 4.4.5 specializes to the claim that $w(8,3) = w(3,1)^2 \cdot w(2,1)$. Indeed, $w(8,3) = xxxyxxxyxxy = (xxxy)^2(xxy) = w(3,1)^2 \cdot w(2,1)$. When we write $w(8,3) = xuy$, we get that $u = xxyxxxyxx$, which is indeed a palindrome, as predicted by Proposition 4.4.6.

4.5 Lee–Schiffler

Given $n \geq 4$, let $\mathcal{D}_n = \mathcal{D}_{(c_{n-2}-c_{n-3})\times c_{n-3}}$. Let $s_{i,j}$ be the slope of the line connecting v_i and v_j. Let $s = s_{0,c_{n-3}}$.

Remark 4.5.1. The indexing of the sequence c_m in Lee and Schiffler (2013) is off by one compared to our sequence c_m. For initial conditions, they have $c_1 = 0$, $c_2 = 1$, whereas we have $c_0 = 0$, $c_1 = 1$. Accordingly, we shall modify their statements to suit our indexing.

Definition 4.5.2 (Lee and Schiffler (2013) Definition 7)**.** For any $0 \leq i < k \leq c_{n-3}$, we define a colored subpath $\alpha(i,k)$ to be the subpath of $\mathcal{D}_n$ defined as follows.

1. If $s_{i,t} \leq s$ for all t such that $i < t \leq k$, then let $\alpha(i,k)$ be the subpath from v_i to v_k. Each of these subpaths will be called a **blue** subpath.

2. If $s_{i,t} > s$ for some $i < t \leq k$, then

   If the smallest such t is of the form $i + c_m - wc_{m-1}$ for some integers $3 \leq m \leq n-2$ and $1 \leq w < r-1$, then let $\alpha(i,k)$ be the subpath from v_i to v_k. Each of these subpaths will be called a green subpath. When m and w are specified, it will be said to be (m,w)-**green**;

   Otherwise, $\alpha(i,k)$ is the subpath from the v_i to v_k. Each of these subpaths will be called a **red** subpath.

Remark 4.5.3. Our definition of a red subpath is slightly different. When $\alpha(i,k)$ is red, Lee and Schiffler (2015) includes the immediate predecessor of v_i as part of the subpath $\alpha(i,k)$, whereas we don't view it as contained in $\alpha(i,k)$, but force it to be included in the definition of $\mathcal{F}(\mathcal{D}_n)$. This will simplify our discussion in Section 4.7.

Let $\mathcal{P}(\mathcal{D}_n)$ be the set of all colored subpaths and single edges:

$$\mathcal{P}(\mathcal{D}_n) = \{\alpha(i,k) \mid 1 \le i < k \le c_{n-3}\} \cup \{\alpha_1, \dots, \alpha_{c_{n-2}}\}.$$

We sometimes call the single edges *colorless* subpaths. We are now ready to introduce the generating set for Lee–Schiffler's combinatorial formula.

Definition 4.5.4 (Definition 8, Lee and Schiffler (2013)).

$$\begin{aligned}\mathcal{F}(\mathcal{D}_n) = \Big\{ &\{\beta_1, \dots, \beta_t\} \mid t \ge 0, \beta_j \in \mathcal{P}(\mathcal{D}_n) \text{ for all } 1 \le j \le t, \\ &\text{if } j \ne j' \text{ then } \beta_j \text{ and } \beta_{j'} \text{ have no common edge;} \\ &\text{if } \beta_j = \alpha(i,k), \beta_{j'} = \alpha(i',k'), \text{ then } i \ne k' \text{ and } i' \ne k; \\ &\text{if } \beta_j \text{ is red, then the edge which immediately precedes } v_i \text{ is contained in some } \beta'_j; \\ &\text{and if } \beta_j \text{ is } (m,w)\text{-green, then at least one of the } (c_{m-1} - wc_{m-2}) \\ &\text{preceding edges of } v_i \text{ is contained in some } \beta_{j'} \Big\}.\end{aligned}$$

Given $\beta \in \mathcal{F}(\mathcal{D}_n)$, let $|\beta|_1 = \sum_{\alpha(i,k)\in\beta}(k-i)$ and let $|\beta|_2$ be the total number of edges in subpaths of β.

Theorem 4.5.5 (Theorem 9, Lee and Schiffler (2013)). For $n \ge 4$,

$$x_n = x_1^{-c_{n-2}} x_2^{-c_{n-3}} \sum_{\beta\in\mathcal{F}(\mathcal{D}_n)} x_1^{r|\beta|_1} x_2^{r(c_{n-2}-|\beta|_2)}.$$

and

$$x_{3-n} = x_2^{-c_{n-2}} x_1^{-c_{n-3}} \sum_{\beta\in\mathcal{F}(\mathcal{D}_n)} x_2^{r|\beta|_1} x_1^{r(c_{n-2}-|\beta|_2)}.$$

It follows that the F-polynomials have the following expansion formula.

Corollary 4.5.6 (Corollary 12, Lee and Schiffler (2013)). Let $n \ge 4$. Then

$$F_n = \sum_{\beta\in\mathcal{F}(\mathcal{D}_n)} y_1^{|\beta|_2} y_2^{|\beta|_1}, \quad F_{3-n} = \sum_{\beta\in\mathcal{F}(\mathcal{D}_n)} y_1^{c_{n-3}-|\beta|_1} y_2^{c_{n-2}-|\beta|_1}.$$

In Section 4.7, we shall also need the set $\widetilde{\mathcal{F}}(\mathcal{D}_n)$, defined below.

Definition 4.5.7.

$$\widetilde{\mathcal{F}}(\mathcal{D}_n) = \Big\{\{\beta_1, \dots, \beta_t\} \mid t \geq 0, \beta_j \in \mathcal{P}(\mathcal{D}_n) \text{ for all } 1 \leq j \leq t,$$
$$\text{if } j \neq j' \text{ then } \beta_j \text{ and } \beta_{j'} \text{ have no common edge,}$$
$$\text{if } \beta_j = \alpha(i,k), \beta_{j'} = \alpha(i',k'), \text{ then } i \neq k' \text{ and } i' \neq k\}.$$

Since the definition of $\widetilde{\mathcal{F}}(\mathcal{D}_n)$ is obtained from that of $\mathcal{F}(\mathcal{D}_n)$ by removing conditions, $\widetilde{\mathcal{F}}(\mathcal{D}_n)$ contains $\mathcal{F}(\mathcal{D}_n)$.

4.5.1 The Lee–Schiffler Formula for the Kronecker quiver

In the case of the Kronecker cluster algebra, Equation 2.1 and Equation 2.2 for cluster variables imply the following formulas for the F-polynomial: for $\ell \geq 1$,

$$F_{\ell+2}(y_1, y_2) = \sum_{M,N \geq 0} \binom{\ell - N}{\ell - M}\binom{M-1}{N} y_1^M y_2^N,$$

and

$$F_{1-\ell}(y_1, y_2) = \sum_{M,N \geq 0} \binom{\ell - N}{\ell - M}\binom{M-1}{N} y_1^{\ell-1-N} y_2^{\ell-M}.$$

As an exercise on Lee and Schiffler (2013), we shall show explicitly in this section that when $r = 2$, their combinatorial formula specializes to the above. Using the fact that $c_\ell = \ell$ when $r = 2$, we specialize Corollary 4.5.6 to the following: for $\ell \geq 2$,

$$F_{\ell+2}(y_1, y_2) = \sum_{\beta \in \mathcal{F}(\mathcal{D}_{\ell+2})} y_1^{|\beta|_2} y_2^{|\beta|_1},$$

and for $\ell \geq 1$,

$$F_{1-\ell}(y_1, y_2) = \sum_{\beta \in \mathcal{F}(\mathcal{D}_{\ell+2})} y_1^{\ell-1-|\beta|_1} y_2^{\ell-|\beta|_2},$$

where $\mathcal{F}(\mathcal{D}_{\ell+2})$ is the collection of non-overlapping families of subpaths from Definition 4.5.4. Note that the maximal Dyck path is simply $\mathcal{D}_{\ell+2} = \mathcal{D}_{(c_\ell - c_{\ell-1}) \times c_{\ell-1}} = \mathcal{D}_{1 \times (\ell-1)}$. The conditions for the color of $\alpha(i,k)$ also simplify significantly. Since there is no integer w such that $1 \leq w < r-1$, by Definition 4.5.2, there can be no green subpaths of $\mathcal{D}_{\ell+2}$. Moreover, since $v_i = (1, i)$ for

$0 < i \leq \ell - 1$, for $0 < i < k \leq \ell - 1$, $s_{i,k}$ is just infinity, which is of course greater than $s = s_{0,\ell-1} = \ell - 1$. Therefore, the subpath $\alpha(i,k)$ is red if $i \neq 0$. We also always have $s_{0,k} \leq s_{0,\ell-1} = s$, which implies that a colored subpath $\alpha(0,k)$ is blue. In either case, $\alpha(i,k)$ consists of $k - i + 1$ edges.

Example 4.5.8. The following calculation illustrates how to apply the Lee–Schiffler formula to obtain the F-polynomials $F_5(y_1, y_2)$ and $F_6(y_1, y_2)$ of the Kronecker cluster algebra. Since we are working with positively indexed F-polynomials, the contribution of an arbitrary $\beta \in \mathcal{F}(\mathcal{D}_{\ell+2})$ is $y_1^{|\beta|_2} y_2^{|\beta|_1}$. In order to enumerate all the subpath families of $\mathcal{F}(\mathcal{D}_{\ell+2})$, it is often beneficial to start by considering which colored subpaths can be chosen together. The set of colored subpaths determines $|\beta|_1$, and then one is free to include or exclude each of the remaining colorless edges, which modifies the total number of edges used in β, namely $|\beta|_2$. For example, for all $\beta \in \mathcal{F}(\mathcal{D}_{1,2})$ such that $\alpha(0,1)$ is its only colored subpath, $|\beta|_1 = 1$, and $|\beta|_2$ is 2 plus the number of colorless edges in β. Since there is only one remaining colorless edge, the total contribution of all such $\beta \in \mathcal{F}(\mathcal{D}_{1\times 2})$ is $y_1^2 y_2(1 + y_1)$. In this manner, we may speak of the contribution of some set of colored subpaths.

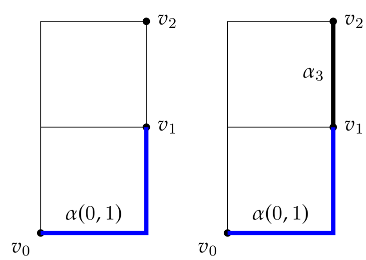
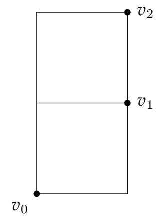


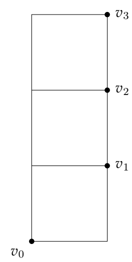
Figure 4.3 Two path families in $\mathcal{F}(\mathcal{D}_{1\times 2})$ whose only colored subpath is $\alpha(0,1)$, whose contributions are respectively $y_1^2 y_2$ (left) and $y_1^3 y_2$ (right)

This is the approach we take to construct Tables 4.2 and 4.3. In each row, we group those choices of colored subpaths that use the same number of edges and have the same $|\beta|_1$, which implies that they contribute in the same way to the final F-polynomial. Indeed, we expect their contribution to be of the form $y_1^A y_2^B (1 + y_1)^{\ell - A}$, where A is the number of edges in the colored subpaths and B is $|\beta|_1$. We use color to indicate whether a colored subpath is blue or red.



Colored Subpaths in β	$\lvert\beta\rvert_1$	#Colorless	Contribution
$\emptyset$	0	3	$(1+y_1)^3$
$\{\alpha(0,1)\}, \{\alpha(1,2)\}$	1	1	$2y_1^2 y_2(1+y_1)$
$\{\alpha(0,2)\}$	2	0	$y_1^3 y_2^2$

Table 4.2 By the Lee–Schiffler Formula, we use $\mathcal{F}(\mathcal{D}_5) = \mathcal{F}(\mathcal{D}_{1\times 2})$ to calculate that $F_5(y_1, y_2) = (1+y_1)^3 + 2y_1^2 y_2(1+y_1) + y_1^3 y_2^2$; the column #Colorless counts the number of remaining colorless edges in $\mathcal{D}_5$



Colored Subpaths in β	$\lvert\beta\rvert_1$	#Colorless	Contribution
$\emptyset$	0	4	$(1+y_1)^4$
$\{\alpha(0,1)\}, \{\alpha(1,2)\}, \{\alpha(2,3)\}$	1	2	$3y_1^2 y_2(1+y_1)^2$
$\{\alpha(0,2)\}, \{\alpha(1,3)\}$	2	1	$2y_1^3 y_2^2(1+y_1)$
$\{\alpha(0,1), \alpha(2,3)\}$	2	0	$y_1^4 y_2^2$
$\{\alpha(0,3)\}$	3	0	$y_1^4 y_2^3$

Table 4.3 By the Lee–Schiffler Formula, we use $\mathcal{F}(\mathcal{D}_6) = \mathcal{F}(\mathcal{D}_{1\times 3})$ to calculate that $F_6(y_1, y_2) = (1+y_1)^4 + 3y_1^2 y_2(1+y_1)^2 + 2y_1^3 y_2^2(1+y_1) + y_1^4 y_2^2 + y_1^4 y_2^3$; the column #Colorless counts the number of remaining colorless edges in $\mathcal{D}_6$

We check that the total number of monomials is 13 in F_5 and 34 in F_6, which is as we expect, since F-polynomials of the Kronecker quiver specialize to every other Fibonacci number.

Let

$$F^{\ell+2}_{M,N} = \left| \beta \in \mathcal{F}(\mathcal{D}_{\ell+2}) : \ |\beta|_1 = N, |\beta|_2 = M \right|,$$
$$C^{\ell+2}_{M,N} = C^{(\ell+2,2)}_{M,N} = \binom{\ell - M}{\ell - N}\binom{M-1}{N}.$$

To reconcile these two formulas, it suffices to prove the following:

Theorem 4.5.9. For $M, N \geq 0$ and $\ell \geq 1$, $F^{\ell+2}_{M,N} = C^{\ell+2}_{M,N}$.

When $N < M \leq \ell$ is not satisfied, $F^{\ell}_{M,N} = 0$ because there are no such $\beta \in \mathcal{F}(\mathcal{D}_{\ell+2})$, and $C^{\ell}_{M,N} = 0$. Therefore it suffices to focus on the cases where $0 \leq N < M \leq \ell$.

By thinking about whether the last vertical edge $\mu_{\ell-1}$ is used and the color of the subpath that uses it, we can write down the following recurrence of the $F^{\ell}_{M,N}$'s:

$$F^{\ell+1}_{M,N} = F^{\ell}_{M,N} + \sum_{k=0}^{\ell} F^{\ell-k}_{M-k-1,N-k}.$$

The term $F^{\ell}_{M,N}$ counts the number of subpath families that don't use the last edge. The term $F^{\ell}_{M-1,N}$ counts the number of those that includes the last edge as a colorless subpath. For $k > 0$, the term $F^{\ell-k}_{M-k-1,N-k}$ counts the number of subpath famimilies that uses the last edge in the colored subpath $\alpha(\ell-k-1, \ell-1)$. Knowing that the Lee–Schiffler formula is correct for small ℓ, it suffices to show that the $C^{\ell}_{M,N}$'s also satisfy the same recurrence relation.

Lemma 4.5.10. When $N < M \le \ell$,

$$\binom{M-1}{N} = \sum_{k=0}^{N} \binom{M-k-2}{N-k} = \sum_{k=0}^{\ell} \binom{M-k-2}{N-k}.$$

Proof. The second equality follows from the fact that $\binom{N}{s} = 0$ if $s < 0$. We prove the first equality by induction on N. This is true when $N = 0$ since both sides are equal to 1. Suppose this is true for $N-1$ and all $M > N-1$. Then

$$\begin{aligned}
\sum_{k=0}^{N} \binom{M-k-2}{N-k} &= \binom{M-2}{N} + \sum_{k=1}^{N} \binom{M-k-2}{N-k} \\
&= \binom{M-2}{N} + \sum_{k=0}^{N-1} \binom{(M-1)-k-2}{(N-1)-k} \\
&= \binom{M-2}{N} + \binom{M-2}{N-1} \\
&= \binom{M-1}{N}
\end{aligned}$$

as desired. □

By Lemma 4.5.10,

$$\binom{\ell-N}{\ell+1-M}\binom{M-1}{N} = \sum_{k=0}^{\ell} \binom{\ell-N}{\ell+1-M}\binom{M-k-2}{N-k}.$$

Rewriting the left hand side as

$$\binom{\ell-N}{\ell+1-M}\binom{M-1}{N}=\binom{\ell+1-N}{\ell+1-M}\binom{M-1}{N}-\binom{\ell-N}{\ell-M}\binom{M-1}{N},$$

we have the identity

$$\binom{\ell+1-N}{\ell+1-M}\binom{M-1}{N}=\binom{\ell-N}{\ell-M}\binom{M-1}{N}+\sum_{k=0}^{\ell}\binom{\ell-N}{\ell+1-M}\binom{M-k-2}{N-k},$$

or equivalently,

$$C_{M,N}^{\ell+1}=C_{M,N}^{\ell}+\sum_{k=0}^{\ell}C_{M-k-1,N-k}^{\ell-k},$$

which is the desired recurrence.

4.6 Lee–Li–Zelevinsky: Greedy Elements

Motivated by the search for "natural" bases in cluster algebras, Lee et al. (2014) studies a collection of *greedy* elements in $\mathcal{A}(b,c)$, which were first introduced in an unpublished follow-up to Sherman and Zelevinsky (2004). The greedy elements are parameterized by $(a_1,a_2)\in\mathbb{Z}^2$, written as $x[a_1,a_2]$. In Lee et al. (2014), the key ingredient for their study of greedy elements is a combinatorial formula using a combinatorial object called *compatible pairs*. This formula specializes to a formula for cluster variables of $\mathcal{A}(b,c)$ because of the following fact.

Theorem 4.6.1 (Theorem 1.7(e), Remark 1.9, Lee et al. (2014))**.** Let $(a_1,a_2)\in\mathbb{Z}^2$ be the **d**-vector of the cluster variable x_m. Then $x_m=x[a_1,a_2]$.

We now introduce compatible pairs, which will lead us to a combinatorial formula for $x[a_1,a_2]$.

Definition 4.6.2 (Definition 1.10, Lee et al. (2014))**.** Let b,c be positive integers. Let $(a_1,a_2)\in\mathbb{Z}^2_{>0}$. Let $\mathrm{Pairs}(a_1,a_2)=\{(S_1,S_2):S_1\subseteq\mathcal{R}_{a_1\times a_2},S_2\subseteq\mathcal{U}_{a_1\times a_2}\}$. Let $(S_1,S_2)\in\mathrm{Pairs}(a_1,a_2)$. We say that the edges (ρ,μ), where $\rho\in S_1$, $\mu\in S_2$, are compatible if, denoting by E the left endpoint of ρ and F the upper endpoint of μ, there exists a lattice point $A\in EF^\circ$ such that

$$|(AF)_1|=b|(AF)_2\cap S_2|\text{ or }|(EA)_1|=c|(EA)_1\cap S_1|.$$

We say that (S_1,S_2) is a *compatible pair* if for every $\rho\in S_1$ and $\mu\in S_2$ are compatible.

Theorem 4.6.3 (Theorem 1.11, Lee et al. (2014))**.** For every $(a_1, a_2) \in \mathbb{Z}^2$, the greedy element $x[a_1, a_2] \in \mathcal{A}(b, c)$ at (a_1, a_2) is given by

$$x[a_1, a_2] = x_1^{-a_1} x_2^{-a_2} \sum_{(S_1, S_2)} x_1^{b|S_2|} x_2^{c|S_1|}, \tag{4.6}$$

where the sum is over all compatible pairs (S_1, S_2) in $\mathcal{D}_{[a_1]_+ \times [a_2]_+}$.

Recall from Section 4.1 that in the cluster algebra $\mathcal{A}(b, c)$, the **d**-vectors of cluster variables are given by

$$\mathbf{d}_{k+2} = \begin{bmatrix} B_k \\ A_{k-1} \end{bmatrix}, \ \mathbf{d}_{-k+1} = \begin{bmatrix} B_{k-1} \\ A_k \end{bmatrix}$$

for $k \geq 0$, where A_k, B_k are integer sequences defined in Section 4.1. In particular, when $b = c = r$, for $k \geq 0$, we have that the **d**-vectors are

$$\mathbf{d}_{k+2} = \begin{bmatrix} c_k \\ c_{k-1} \end{bmatrix}, \ \mathbf{d}_{-k+1} = \begin{bmatrix} c_{k-1} \\ c_k \end{bmatrix}.$$

Corollary 4.6.4. For $k \geq 0$, let $\mathcal{C}_{k+2}$ denote the set of compatible pairs in $\mathcal{D}_{[c_k]_+ \times [c_{k-1}]_+}$, and let $\mathcal{C}_{-k+1}$ denote the set of compatible pairs in $\mathcal{D}_{[c_{k-1}]_+ \times [c_k]_+}$. Then $x_m \in \mathcal{A}(r, r)$ is given by

$$x_m = \mathbf{x}^{-\mathbf{d}_m} \sum_{(S_1, S_2) \in \mathcal{C}_m} x_1^{r|S_2|} x_2^{r|S_1|}.$$

Let us focus on the case where $b = c = r$, $a_1 = c_{n-2}$, $a_2 = c_{n-3}$ for some $n \geq 4$, which is the important case for the next section. Using shadows and remote shadows, we can slightly simplify the check for whether $(S_1, S_2) \in \mathrm{Pairs}(a_1, a_2)$ is a compatible pair.

Definition 4.6.5 (Definition 3.6 and 3.7, Lee et al. (2014))**.** For every vertical edge $\mu \in S_2$ with upper endpoint F, let the *local shadow of* S_2 *at* μ, denoted $\mathrm{sh}(\mu; S_2)$, be the set of horizontal edges $(AF)_1$ in the shortest subpath AF of $\mathcal{D}_{a_1 \times a_2}$ such that $|(AF)_1| = r|(AF)_2 \cap S_2|$. If there is no such subpath AF, we define $\mathrm{sh}(\mu; S_2)$ as $(FF)_1 = \mathcal{R}_{a_1 \times a_2}$. For $S \subseteq S_2$, let $\mathrm{sh}(S; S_2) = \cup_{\mu \in S} \mathrm{sh}(\mu; S_2)$, and write $\mathrm{sh}(S_2) := \mathrm{sh}(S_2; S_2)$.

There are either $r - 1$ or r horizontal edges at any height in the maximal Dyck path $\mathcal{D}_{a_1 \times a_2}$, and so the local shadow $\mathrm{sh}(\mu_j; S_2)$ always contains all the horizontal edges in $\mathcal{D}_{a_1 \times a_2}$ of height $j - 1$. For all $\mu_j \in S_2$, remove the horizontal edges of height $j-1$ from $\mathrm{sh}(S; S_2)$. The set of remaining horizontal edges is the *remote shadow of* S *with respect to* S_2, denoted $\mathrm{rsh}(S; S_2)$. We write $\mathrm{rsh}(S_2) := \mathrm{rsh}(S_2; S_2)$.

The following lemma then gives an equivalent condition for compatibility, which we state in the generality of $\mathrm{Pairs}(c_{n-2}, c_{n-3})$.

Lemma 4.6.6 (Lemma 3.4, Lee et al. (2014))**.** Let $n \geq 4$. Let $(S_1, S_2) \in \mathrm{Pairs}(c_{n-2}, c_{n-3})$. Then (S_1, S_2) is a compatible pair if and only if $S_1 \cap (\mathrm{sh}(S_2) - \mathrm{rsh}(S_2)) = \emptyset$ and $(S_1 \cap \mathrm{rsh}(S_2), S_2)$ is compatible.

This lemma suggests the following pseudo-compatibility condition.

Definition 4.6.7. Let $(S_1, S_2) \in \mathrm{Pairs}(c_{n-2}, c_{n-3})$. Then (S_1, S_2) is a *pseudo-compatible pair* if $S_1 \cap (\mathrm{sh}(S_2) \setminus \mathrm{rsh}(S_2)) = \emptyset$. We denote the set of pseudo-compatible pairs in $\mathrm{Pairs}(c_{n-2}, c_{n-3})$ by $\widetilde{C}_n$. Clearly, $C_n \subseteq \widetilde{C}_n$.

The following lemma allows us to speak of maximal remote shadows.

Lemma 4.6.8 (Lemma 3.10, Lee et al. (2014))**.** If μ and μ' are distinct vertical edges from S_2, and both local shadows $\mathrm{sh}(\mu; S_2)$ and $\mathrm{sh}(\mu'; S_2)$ are different from $\mathcal{R}$, then either these local shadows are disjoint, or one of them is a proper subset of another.

We include examples of remote shadows and illustrations of these lemmas in Section 4.7.

4.7 Lee–Schiffler versus Lee–Li–Zelevinsky: Statement of Conjectural Bijection

In this section, towards reconciling the formulas of Lee–Schiffler and Lee–Li–Zelevinsky for cluster variables in $\mathcal{A}(r, r)$, we introduce a conjectural weight-preserving bijection between certain colored subpath families and certain compatible pairs.

Remark 4.7.1. When stating the formula for x_n, Lee and Schiffler (2013) divides into two cases: $n \geq 4$ and $n \leq -1$, whereas Corollary 4.6.4 divides into $n \geq 2$ and $n \leq 1$. One notices that Lee and Schiffler (2013) does not cover the initial variables x_1, x_2, nor the adjacent x_3 and x_0. This perhaps comes from the inconvenience that these are exactly the four cluster variables whose **d**-vectors involve 0, and as such, involve trivial maximal Dyck paths. In what ensues, we shall attempt to reconcile the two formulas for x_n where $n \geq 4$ or $n \leq -1$.

Let us now recall the formulas of Lee and Schiffler (2013) and Lee et al. (2014) for cluster variables of $\mathcal{A}(r,r)$. We introduce the notion of weights to simplify our notation. Instead of defining weights on the sets $\mathcal{F}(\mathcal{D}_n)$ and C_n, we do so on the sets $\widetilde{\mathcal{F}}(\mathcal{D}_n)$ and $\widetilde{C}_n$.

Definition 4.7.2. Let $n \geq 4$ or $n \leq -1$. For $\beta \in \tilde{\mathcal{F}}(\mathcal{D}_n)$, let

$$\mathrm{wt}_n(\beta) = \begin{cases} x_1^{r|\beta|_1} x_2^{r(c_{n-2}-|\beta|_2)} & n \geq 4, \\ x_2^{r|\beta|_1} x_1^{r(c_{n-2}-|\beta|_2)} & n \leq -1. \end{cases}$$

For $(S_1, S_2) \in \widetilde{C}_n$, let $\mathrm{wt}(S_1, S_2) = x_1^{r|S_2|} x_2^{r|S_1|}$.

When $n \geq 4$, rephrased in terms of weights, Lee and Schiffler (2013) says that

$$x_n = x_1^{-c_{n-2}} x_2^{-c_{n-3}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} \mathrm{wt}_n(\beta),$$

where $\mathcal{F}(\mathcal{D}_n)$ is the set of families of colored subpaths of $\mathcal{D}_{(c_{n-2}-c_{n-3})\times c_{n-3}}$. On the other hand, Lee et al. (2014) says that

$$x_n = x_1^{-c_{n-2}} x_2^{-c_{n-3}} \sum_{(S_1,S_2)\in C_n} \mathrm{wt}(S_1, S_2),$$

where C_n is the set of compatible pairs in $\mathcal{D}_{c_{n-2}\times c_{n-3}}$. (We may discard the $[\cdot]_+$ from Corollary 4.6.4 because $c_n > 0$ for $n \geq 1$.)

Comparing Lee–Schiffler's and Lee–Li–Zelevinsky's formulas for $n \geq 4$, we see that to show that these formulas are equivalent, it suffices to find a bijection $\Phi = (\Phi_1, \Phi_2) : \mathcal{F}(\mathcal{D}_n) \to C_n$, such that $\mathrm{wt}(\Phi(\beta)) = \mathrm{wt}_n(\beta)$. Comparing $\mathrm{wt}(\Phi(\beta)) = x_1^{r|S_2|} x_2^{r|S_1|}$ and $\mathrm{wt}_n(\beta) = x_1^{r|\beta|_1} x_2^{r(c_{n-2}-|\beta|_2)}$, we see that the weight-preserving condition is equivalent to

$$|\Phi_1(\beta)| = c_{n-2} - |\beta|_2, \; |\Phi_2(\beta)| = |\beta|_1.$$

We now describe a weight-preserving bijection $\widetilde{\Phi} : \widetilde{\mathcal{F}}(\mathcal{D}_n) \to \widetilde{C}_n$, which conjecturally restricts to a bijection between $\mathcal{F}(\mathcal{D}_n)$ and C_n. Given $\beta = \{\beta_1, \ldots, \beta_m\} \in \widetilde{\mathcal{F}}_n$, let $\{\alpha(i_1, k_1), \ldots, \alpha(i_t, k_t)\}$ be the set of colors subpaths in β, where $0 \leq i_1 < k_1 < i_2 < \cdots < i_t < k_t \leq c_{n-3}$. Then let

$$\widetilde{\Phi}_1(\beta) = \{\rho_s : \alpha_s \notin \beta_i \text{ for any } 1 \leq i \leq m\},$$
$$\widetilde{\Phi}_2(\beta) = \{\mu_s : i_j < s \leq k_j \text{ for some } 1 \leq j \leq t\},$$

and let $\widetilde{\Phi}(\beta) = (\widetilde{\Phi}_1(\beta), \widetilde{\Phi}_2(\beta))$.

Proposition 4.7.3. Let $n \geq 4$. The function $\widetilde{\Phi}$ described above is indeed a weight-preserving bijection between $\widetilde{\mathcal{F}}(\mathcal{D}_n)$ and $\widetilde{C}_n$.

Proof. As a function into $\mathrm{Pairs}(c_{n-2}, c_{n-3})$, $\widetilde{\Phi}$ is immediately weight-preserving and injective.

Notice that $\frac{m}{n} \leq \frac{a}{b}$ if and only if $\frac{m}{m+n} \leq \frac{a}{a+b}$. It follows that $\mathcal{D}_{c_{n-2},c_{n-3}}$ can be obtained from $\mathcal{D}_n = \mathcal{D}_{c_{n-2}-c_{n-3},c_{n-2}}$ by adding a horizontal edge before every vertical edge. This gives rise to a natural bijection $\mathcal{E}_{(c_{n-2}-c_{n-3})\times c_{n-3}} \cong \mathcal{R}_{c_{n-2}\times c_{n-3}}$.

Now, if $\mu_j \in \widetilde{\Phi}_2(\beta)$ for some $\beta \in \widetilde{\mathcal{F}}(\mathcal{D}_n)$, then as an edge of $\mathcal{D}_n$, μ_j is contained in some colored $\alpha(i_s, k_s) \in \beta$. Under the above correspondence, the horizontal edges of height $j-1$ in $\mathcal{D}_{c_{n-2},c_{n-3}}$ correspond in $\mathcal{D}_n$ to the horizontal edges of height $j-1$ plus the vertical edge μ_j. Since all horizontal edges of height $j-1$ in $\mathcal{D}_n$ as well as μ_j are used in $\alpha(i_s, k_s)$, none of the horizontal edges of height $j-1$ in $\mathcal{D}_{c_{n-2},c_{n-3}}$ can be included in $\widetilde{\Phi}_1(\beta)$. Therefore, $\widetilde{\Phi}(\beta) \in \widetilde{C}_n$ for all $\beta \in \widetilde{\mathcal{F}}(\mathcal{D}_n)$.

We now show surjectivity. Given a pair $(S_1, S_2) \in \widetilde{C}_n$, a preimage in $\widetilde{\mathcal{F}}(\mathcal{D}_n)$ exists if and only if for every vertical edge μ_j in S_2, the set $A = \{\alpha_s : \rho_s \notin S_1\}$ of edges in $\mathcal{D}_n$ contains μ_j and all horizontal edges at height $j-1$. This follows from the fact that S_1 does not contain any of the horizontal edges of height $j-1$. $\square$

Conjecture 3. The map $\widetilde{\Phi} : \widetilde{\mathcal{F}}(\mathcal{D}_n) \to \widetilde{C}_n$ restricts to a bijection $\Phi : \mathcal{F}(\mathcal{D}_n) \to C_n$.

Let us consider two examples of $\widetilde{\Phi}$ applied to $\beta \in \mathcal{F}(\mathcal{D}_n)$.

Example 4.7.4. Let $r = 3$. We will be considering the bijection $\Phi : \mathcal{F}(\mathcal{D}_7) \to C_7$. Recall that $\mathcal{D}_7 = \mathcal{D}_{34\times 21}$, and C_7 consists of compatible pairs in $R_{55\times 21}$. Let $\beta_1 = \{\alpha(8, 21), \alpha_{19}\}$, pictured in Figure 4.4, and let $\beta_2 = \{\alpha(3, 7), \alpha(8, 21)\}$, which is pictured in Figure 4.6. Since $\alpha(8, 21)$ is green, and since $21 - 8 = c_4 - c_3$, by Definition 4.5.4, one of the $c_3 - c_2 = 8 - 3 = 5$ edges right before v_8 must be used. In β_1, the colorless edge α_{19} fulfills this requirement, whereas in β_2, the blue path $\alpha(3, 7)$ fulfills this requirement.

We check a few pairs of edges to convince ourselves that $\Phi(\beta_1), \Phi(\beta_2))$ are indeed both compatible pairs. Consider the pair (ρ_{17}, μ_{21}) in $\Phi(\beta_1)$, labeled in Figure 4.5. Let E be the left endpoint of ρ_{17} and let $F = v_{21}$. The horizontal edge ρ_{17} is in $\mathrm{rsh}(S_2)$ because $|(EF)_1| = 39 = 3|(EF)_2 \cap \Phi_2(\beta_1)|$, and E is such that EF is the shortest such subpath. Therefore, to show that

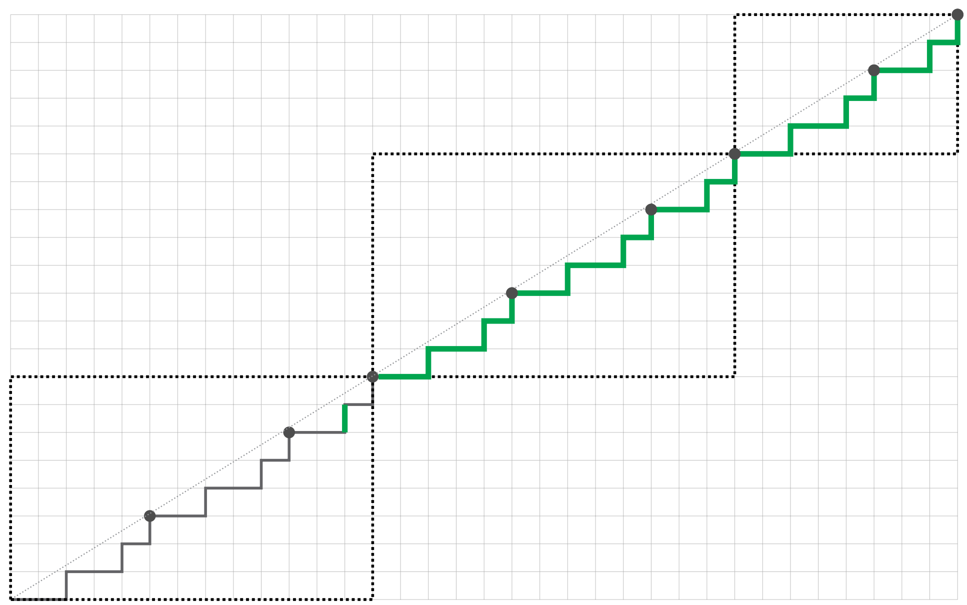


Figure 4.4 $\beta_1 = \{\alpha_{19}, \alpha(8, 21)\}$ in $R_{34\times 21}$; the paths used in β are bolded in green

the edges ρ_{17}, μ_{21} are compatible, we can only try to find $A \in (EF)^\circ$ such that

$$|(EA)_2| = 3|(EA)_1 \cap \Phi_1(\beta_1)|.$$

This is satisfied by $A = v_{18}$.

Notice that if we include all of the five preceding horizontal edges of v_8, then choosing any A which comes after v_8 would result in $|(EA)_1 \cap \Phi_1(\beta_2)| = 5$, which would require us to find $A \in (EF)^\circ$ such that $|(EA)_2| = 15$. However, even if we chose $A = v_{20}$, we can only get $|(EA)_2| = 14$. This demonstrates the necessity of the conditions which distinguish $\mathcal{F}(\mathcal{D}_n)$ from $\widetilde{\mathcal{F}}(\mathcal{D}_n)$.

Let us now consider the compatibility of the edges ρ_5, μ_{21} in the second example. Again let E be the left endpoint of ρ_5 and let $F = v_{21}$. We may check again that $\rho_5 \in \mathrm{sh}(\mu_{21}; \Phi_2(\beta_2))$. However, we can choose $A = v_{19}$, which gives us $|(EA)_2| = 18 = 3|(EA)_1 \cap \Phi_1(\beta_2)|$. Similar to before, if we had included, say ρ_{19}, then that would require us to find $A \in (EF)^\circ$ such that $|(EA)_2| = 21$, which is impossible.

In the remainder of this section, we will discuss why Conjecture 3 also suffices to prove that Lee–Schiffler's and Lee–Li–Zelevinsky's formulas for x_n are equivalent when $n \leq -1$. Let us compare their formulas for the $n \leq -1$

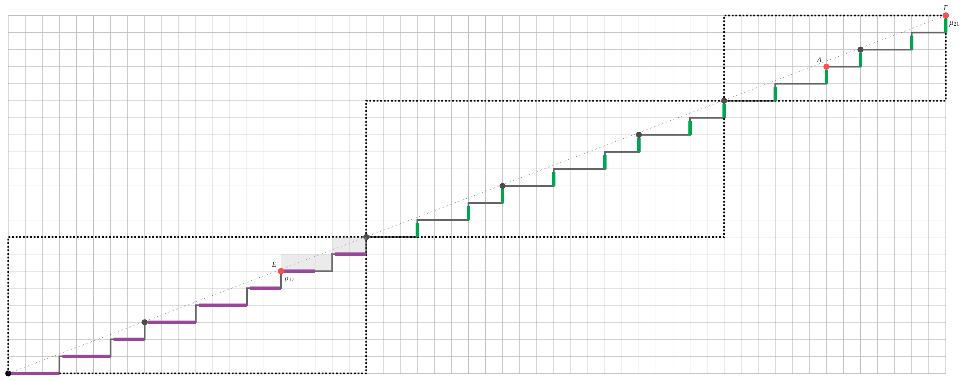


Figure 4.5 $\Phi(\beta_1)$ in $R_{55\times21}$; the edges in $\Phi_1(\beta_1)$ are in purple and the edges in $\Phi_2(\beta_1)$ are in green

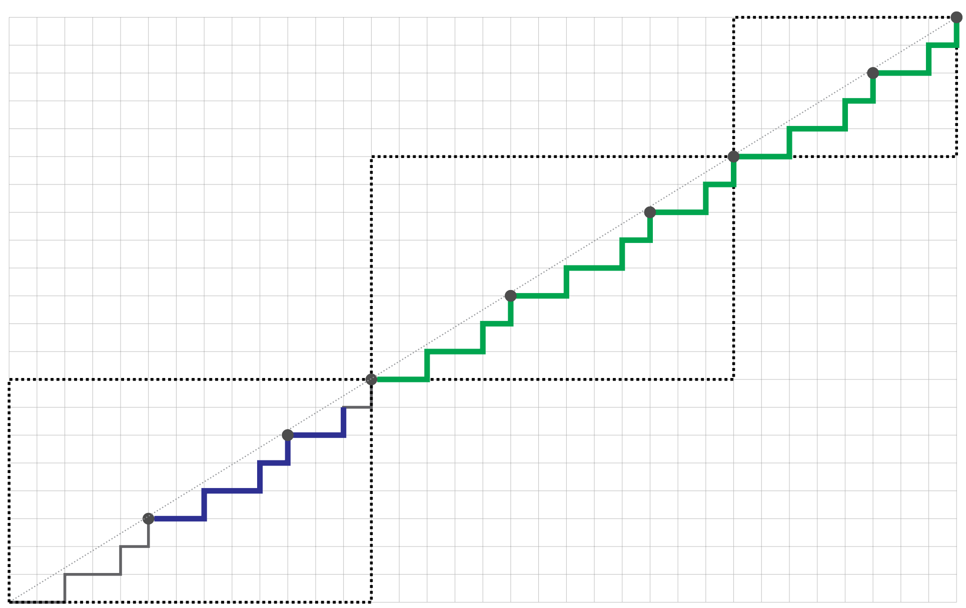


Figure 4.6 $\beta_2 = \{\alpha(3,7), \alpha(8,21)\}$ in $R_{34\times21}$

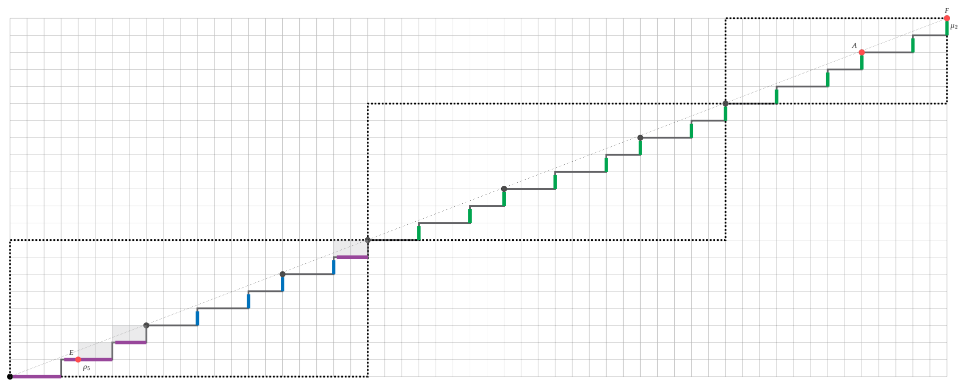


Figure 4.7 $\Phi(\beta_2)$ in $R_{55\times 21}$; the edges in $\Phi_1(\beta_2)$ are in purple and the edges in $\Phi_2(\beta_2)$ are in blue or green according to the color of the subpath of $\mathcal{D}_{34\times 21}$ which contains them

case. Lee and Schiffler (2013) says that for $n \geq 4$,

$$x_{3-n} = x_2^{-c_{n-2}} x_1^{-c_{n-3}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} \mathrm{wt}_{3-n}(\beta),$$

where $\mathcal{F}(\mathcal{D}_n)$ is the same set of families of colored subpaths of $\mathcal{D}_{(c_{n-2}-c_{n-3})\times c_{n-3}}$. Lee et al. (2014) says that

$$x_{3-n} = x_1^{-c_{n-3}} x_2^{-c_{n-2}} \sum_{(S_1,S_2)\in C_{3-n}} \mathrm{wt}(S_1, S_2),$$

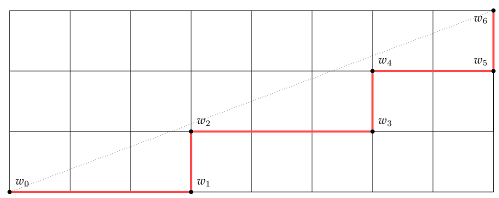
where C_{3-n} is the set of compatible pairs in $\mathcal{D}_{c_{n-3}\times c_{n-2}}$. Comparing Lee–Schiffler's and Lee–Li–Zelevinsky's formulas for $n \leq -1$, we see that to show that these formulas are equivalent, it suffices to find a bijection $\widetilde{\Phi} = (\widetilde{\Phi}_1, \widetilde{\Phi}_2) : \mathcal{F}(\mathcal{D}_n) \to C_{3-n}$ such that $\mathrm{wt}_{3-n}(\beta) = \mathrm{wt}(\widetilde{\Phi}(\beta))$; in other words, we need

$$|\widetilde{\Phi}_1(\beta)| = |\beta|_1, \;\; |\widetilde{\Phi}_2(\beta)| = c_{n-2} - |\beta|_2.$$

Assuming Conjecture 3, it suffices to find a bijection $\gamma = (\gamma_1, \gamma_2) : C_n \to C_{3-n}$ such that

$$|\gamma_1(S_1, S_2)| = |S_2|, \;\; |\gamma_2(S_1, S_2)| = |S_1|. \tag{4.7}$$

To motivate our definition, let us consider the specific case when $n = 5$ and $r = 3$. Recall that C_5 and C_{-2} are the sets of compatible pairs in $\mathcal{D}_{c_3\times c_2} = \mathcal{D}_{8\times 3}$ and $\mathcal{D}_{c_2\times c_3} = \mathcal{D}_{3\times 8}$. From Figure 4.8, we notice that $\mathcal{D}_{3\times 8}$ can be obtained from $\mathcal{D}_{8\times 3}$ after a rotation and a reflection. This is captured rigorously by the following lemma.



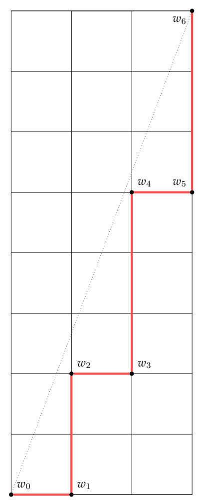


Figure 4.8 $\mathcal{D}_{8\times 3} = xxxyxxxyxxy$ and $\mathcal{D}_{3\times 8} = xyyxyyyxyyy$

Lemma 4.7.5. Let a, b be positive integers. Let a transposition be the operation on $\{x, y\}$-words which replaces all x's by y's and all y's by x's. Then the word $w(b, a)$ can be obtained by transposing the reversal of $w(a, b)$.

Proof. Recall from Definition 4.4.1 that the word $w(a, b)$ can be defined by the property that the number of y's in the first k letters is equal to $\left\lfloor \frac{bk}{a+b} \right\rfloor$.

Let $\widetilde{w}$ be the transpose of the reversal of $w(a, b)$. The number of y's in the first k letters of $\widetilde{w}$ is equal to a minus the number of x's in the first $a + b - k$ letters of w, and the latter is equal to

$$a - \left(a + b - k - \left\lfloor \frac{b(a + b - k)}{a + b} \right\rfloor\right) = \left\lfloor \frac{ak}{a + b} \right\rfloor.$$

This shows that $\widetilde{w} = w(b, a)$. □

This lemma provides natural bijections

$$\mathcal{R}_{a\times b} \to \mathcal{U}_{b\times a}, \ \mathcal{U}_{a\times b} \to \mathcal{R}_{b\times a}$$

defined by sending ρ_k to μ_{a-k} and μ_k to ρ_{b-k}. This bijection induces a map between compatible pairs.

Lemma 4.7.6. Let $n \geq 4$. Let

$$\gamma' = (\gamma'_1, \gamma'_2) : \text{Pairs}(c_{n-2}, c_{n-3}) \to \text{Pairs}(c_{n-3}, c_{n-2})$$

be the function induced by the bijections

$$\mathcal{R}_{c_{n-2}\times c_{n-3}} \to \mathcal{U}_{c_{n-3}\times c_{n-2}},\ \mathcal{U}_{c_{n-2}\times c_{n-3}} \to \mathcal{R}_{c_{n-3}\times c_{n-2}}$$

described above. Then γ' restricts to a bijection $\gamma : C_n \to C_{3-n}$ that satisfies Equation 4.7.

Proof. By construction, the function γ' is a bijection and satisfies Equation 4.7. It suffices to show that when (S_1, S_2) is a compatible pair, $\gamma'(S_1, S_2)$ is also a compatible pair. This follows from Lemma 4.7.5 and the symmetry of Definition 4.6.2. □

4.8 Progress towards Conjecture 3

In this section, by default, we consider compatible pairs in Pairs(c_{n-2}, c_{n-3}) and use the shorthand notations $\mathcal{U} = \mathcal{U}_{c_{n-2}\times c_{n-3}}$, $\mathcal{R} = \mathcal{R}_{c_{n-2}\times c_{n-3}}$.

Lemma 4.6.8 motivates a reduction to irreducible vertical edge sets, defined below.

Definition 4.8.1. Let $A \subseteq S_2 \subseteq \mathcal{U}$. If there exists $\mu \in S_2$ such that $\mathrm{sh}(\mu'; S_2) \subseteq \mathrm{sh}(\mu; S_2)$ for all $\mu' \in A$, and $\mathrm{sh}(\mu'; S_2) \cap \mathrm{sh}(\mu; S_2) = \emptyset$ for all $\mu' \in S_2 \setminus A$, we say that A is *irreducible with respect to* S_2. We say that S_2 is *irreducible* if it is irreducible with respect to itself.

The following proposition allows us to use irreducibility and relative irreducibility interchangeably in the context of decomposing some S_2 into irreducible subsets.

Proposition 4.8.2. Every $S_2 \subseteq \mathcal{U}$ can be written uniquely as a union of subsets which are irreducible with respect to it. If $A \subseteq S_2$ is irreducible with respect to S_2, then $\mathrm{sh}(\mu; S_2) = \mathrm{sh}(\mu; A)$ for all $\mu \in A$. In particular, A is irreducible and $\mathrm{sh}(A; S_2) = \mathrm{sh}(A)$.

Proof. By Lemma 4.6.8, for any $S_2 \subseteq \mathcal{U}$, there exists $\mu_1, \ldots, \mu_k$ such that the local shadows $\mathrm{sh}(\mu_j; S_2)$ are disjoint and $\mathrm{sh}(S_2) = \bigcup_{j=1}^{k} \mathrm{sh}(\mu_j; S_2)$. This gives a natural partition of S_2 into $A_1, \ldots, A_k$ such that μ_j is the top edge of A_j.

It remains to prove that if $\mu \in A_j$, then $\mathrm{sh}(\mu; A_j) = \mathrm{sh}(\mu; S_2)$, which implies that $\mathrm{sh}(A_j) = \mathrm{sh}(A_j; S_2)$. Since $A_j \subseteq S_2$, we must have $\mathrm{sh}(\mu; A_j) \subseteq \mathrm{sh}(\mu; S_2)$. Now suppose that the top endpoint of μ is F, and AF is the shortest subpath such that $|(AF)_1| = r|(AF)_2 \cap S_2|$. We claim that $(AF)_2 \cap S_2 = (AF)_2 \cap$

A_j. Otherwise, there exists $\mu' \in A_{j'} \neq A_j$ such that $\mu' \in (AF)_2$. But that means $\mathrm{sh}(\mu'; S_2) \cap \mathrm{sh}(\mu; S_2) \neq \emptyset$, which implies that $\mathrm{sh}(A_{j'}; S_2) \cap \mathrm{sh}(A_j; S_2) \neq \emptyset$, which is a contradiction. Therefore, AF is also the shortest subpath such that $|(AF)_1| = r|(AF)_2 \cap A_j|$. $\square$

Example 4.8.3. Both $\Phi_2(\beta_1)$ and $\Phi_2(\beta_2)$), pictured in Figures 4.5 and 4.7, are irreducible. Indeed, $\mathrm{sh}(\Phi_2(\beta_i)) = \mathrm{sh}(v_{21}; \Phi_2(\beta_i))$ for $i = 1, 2$.

Using these ideas, we reduce Conjecture 3 to the following.

Conjecture 4. Let $(S_1, S_2) \in \widetilde{\mathcal{C}}_n$ such that $S_2 \neq \emptyset$ is irreducible and $S_1 \subseteq \mathrm{rsh}(S_2)$. Then $(S_1, S_2) \in \mathcal{C}_n$ if and only if $\widetilde{\Phi}^{-1}(S_1, S_2) \in \mathcal{F}(\mathcal{D}_n)$.

Proposition 4.8.4. Conjecture 4 implies Conjecture 3.

Proof. When $S_2 = \emptyset$, we always have $(S_1, \emptyset) \in \mathcal{C}_n$. Notice also that $\widetilde{\Phi}^{-1}(S_1, \emptyset) = \{\alpha_s : \rho_s \notin S_1\}$, which is just a set of colorless edges, and so is also always in $\mathcal{F}(\mathcal{D}_n)$. So in this case, trivially, we have that $(S_1, \emptyset) \in \mathcal{C}_n$ if and only if $\widetilde{\Phi}^{-1}(S_1, \emptyset) \in \mathcal{F}(\mathcal{D}_n)$.

Now suppose that $(S_1, S_2) \in \widetilde{\mathcal{C}}_n$ is such that $S_2 \neq \emptyset$ and $S_2 = A \sqcup B$ where A and B are nonempty, irreducible and $\mathrm{sh}(A) \cap \mathrm{sh}(B) = \emptyset$. Let $S_{1,A} = S_1 \cap \mathrm{rsh}(A)$, $S_{1,B} = S_1 \cap \mathrm{rsh}(B)$. Assuming that

$$\widetilde{\Phi}^{-1}(S_{1,A}, A) \in \mathcal{F}(\mathcal{D}_n) \Leftrightarrow (S_{1,A}, A) \in \mathcal{C}_n$$

and

$$\widetilde{\Phi}^{-1}(S_{1,B}, B) \in \mathcal{F}(\mathcal{D}_n) \Leftrightarrow (S_{1,B}, B) \in \mathcal{C}_n,$$

we would like to show that $\widetilde{\Phi}^{-1}(S_1, S_2) \in \mathcal{F}(\mathcal{D}_n) \Leftrightarrow (S_1, S_2) \in \mathcal{C}_n$. So we need to show the following:

- $\widetilde{\Phi}^{-1}(S_1, S_2) \in \mathcal{F}(\mathcal{D}_n)$ if and only if $\widetilde{\Phi}^{-1}(S_{1,A}, A) \in \mathcal{F}(\mathcal{D}_n)$ and $\widetilde{\Phi}^{-1}(S_{1,B}, B) \in \mathcal{F}(\mathcal{D}_n)$;
- $(S_1, S_2) \in \mathcal{C}_n$ if and only if $(S_{1,A}, A) \in \mathcal{C}_n$ and $(S_{1,B}, B) \in \mathcal{C}_n$.

The key fact that we need to rely on is that if $\alpha_j \in \widetilde{\Phi}^{-1}(S_1, S_2)$ is one of the immediately preceding $c_{m-1} - wc_{m-2}$ edges of some (m, w)-green $\alpha(i, t) \in \widetilde{\Phi}^{-1}(S_1, S_2)$, or if it is the immediately preceding edge of some red $\alpha(i, t) \in \widetilde{\Phi}^{-1}(S_1, S_2)$, then $\rho_j \in \mathrm{rsh}(S_2)$. Recall the definition of j_o from Lemma 4.8.12. It is clear that $\rho_j \in \mathrm{sh}(\mu_{j_o}; S_2) \subseteq \mathrm{sh}(S_2)$. Since α_j is a colorless edge of $\widetilde{\Phi}^{-1}(S_1, S_2)$, it is disjoint from any colored paths. Therefore, $\rho_j \in \mathrm{rsh}(S_2)$.

Now let us prove the first item. Suppose that $\widetilde{\Phi}^{-1}(S_1, S_2) \in \mathcal{F}(\mathcal{D}_n)$. For any $\alpha(i,t) \in \widetilde{\Phi}^{-1}(S_{1,A}, A)$ which is green or red, we want to show that one of the preceding edges, as required by the definition of $\mathcal{F}(\mathcal{D}_n)$, has been included. We know that there exists some $\alpha_j \in \widetilde{\Phi}^{-1}(S_1, S_2)$ or $\alpha(i',t') \in \widetilde{\Phi}^{-1}(S_1, S_2)$ which satisfies this requirement. If it was some colorless edge $\alpha_j \in \widetilde{\Phi}^{-1}(S_1, S_2)$ which satisfied this requirement, we have $\rho_j \notin S_1$, which implies that $\rho_j \notin S_1 \cap \mathrm{rsh}(A) = S_{1,A}$, and so α_j is used in $\widetilde{\Phi}^{-1}(S_{1,A}, A)$. If it was a colored subpath $\alpha(i',t') \in \widetilde{\Phi}^{-1}(S_1, S_2)$ which satisfies this requirement, then we know that ρ_j lies in the intersection of the shadows of the vertical edges in $\alpha(i,t)$ and the vertical edges in $\alpha(i',t')$. Since A is irreducible, it must be the case that $\alpha(i',t') \in \widetilde{\Phi}^{-1}(S_{1,A}, A)$. Therefore, $\widetilde{\Phi}^{-1}(S_{1,A}, A) \in \mathcal{F}(\mathcal{D}_n)$. The proof that $\widetilde{\Phi}^{-1}(S_{1,B}, B) \in \mathcal{F}(\mathcal{D}_n)$ is analogous. This shows the "only if" direction of the first claim.

Now suppose that $\widetilde{\Phi}^{-1}(S_{1,A}, A) \in \mathcal{F}(\mathcal{D}_n)$ and $\widetilde{\Phi}^{-1}(S_{1,B}, B) \in \mathcal{F}(\mathcal{D}_n)$. For any $\alpha(i,t) \in \widetilde{\Phi}^{-1}(S_1, S_2)$ which is green or red, we want to show that one of the preceding edges, as required by the definition of $\mathcal{F}(\mathcal{D}_n)$, has been included. By definition of irreducibility, the vertical edges of $\alpha(i,t)$ must all be contained in either A or B. Without loss of generality, suppose that the vertical edges of $\alpha(i,t)$ are contained in A. Then since $\widetilde{\Phi}^{-1}(S_{1,A}, A) \in \mathcal{F}(\mathcal{D}_n)$, there exists $\alpha_j \in \widetilde{\Phi}^{-1}(S_{1,A}, A)$ or $\alpha(i',t') \in \widetilde{\Phi}^{-1}(S_{1,A}, A)$ which satisfies this requirement. If the requirement is satisfied by $\alpha_j \in \widetilde{\Phi}^{-1}(S_{1,A}, A)$, then we have $\rho_j \notin S_{1,A} = S_1 \cap \mathrm{rsh}(S_2)$. But by the key fact, $\rho_j \in \mathrm{rsh}(A) \subseteq \mathrm{rsh}(S_2)$. So we must have $\rho_j \notin S_1$, which implies that $\alpha_j \in \widetilde{\Phi}^{-1}(S_1, S_2)$. On the other hand, if the requirement is satisfied by $\alpha(i',t') \in \widetilde{\Phi}^{-1}(S_{1,A}, A)$, for an arbitrary edge α_j in $\alpha(i,t)$, we have that $\rho_j \in \mathrm{sh}(A) \setminus \mathrm{rsh}(A)$, which implies that $\rho \in \mathrm{sh}(S_2) \setminus \mathrm{rsh}(S_2)$. Since $(S_1, S_2) \in \widetilde{C}_n$, we can conclude that $\rho_j \notin S_1$. In particular, $\rho_j \notin \mathrm{rsh}(S_2)$. Since this is true for all $\alpha_j \in \alpha(i',t')$, we have that $\alpha(i',t') \in \widetilde{\Phi}^{-1}(S_1, S_2)$. Therefore, $\widetilde{\Phi}^{-1}(S_1, S_2) \in \mathcal{F}(\mathcal{D}_n)$. This shows the "if" direction, which concludes the proof of the first claim.

Now let us proceed to the second claim. Consider a pair (ρ, μ), where, without loss of generality, $\mu \in S_2$ is a vertical edge in A, $\rho \in S_1$ is a horizontal edge in $\mathrm{sh}(\mu; S_2) = \mathrm{sh}(\mu; A)$ (the equality follows from Proposition 4.8.2). Since $(S_1, S_2) \in \widetilde{C}_n$, Let E be the left endpoint of ρ and let G be the upper endpoint of μ. We wish to show that there exists $F \in (EG)^\circ$ which demonstrates the compatibility of (ρ, μ) with respect to (S_1, S_2) if and only if there exists $F \in (EG)^\circ$ which demonstrates the compatibility of (ρ, μ) with respect to $(S_{1,A}, A)$. It suffices to show that for any $F \in (EG)^\circ$,

$(EF)_1 \cap S_1 = (EF)_1 \cap S_{1,A}$. Since $S_{1,A} = S_1 \cap \mathrm{rsh}(A)$, it suffices to show that $(EF)_1 \cap S_1 \subseteq \mathrm{rsh}(A)$. By definition of local shadows, the local shadow of any vertical edge consists consecutive horizontal edges, which implies that $(EF)_1 \subseteq \mathrm{sh}(\mu; S_2) \subseteq \mathrm{sh}(A; S_2) = \mathrm{sh}(A)$, where the last equality follows from Proposition 4.8.2. Since $S_1 \cap (\mathrm{sh}(A) \setminus \mathrm{rsh}(A)) = \emptyset$, $(EF)_1 \cap S_1 \subseteq \mathrm{rsh}(A)$, as desired.

The general case where S_2 is a disjoint union of more than two irreducible parts is then covered by induction.

We may also easily reduce to the case where $S_1 \subseteq \mathrm{rsh}(S_2)$ using Lemma 4.6.6. □

Given a pair $(S_1, S_2) \in \mathrm{Pairs}(c_{n-2}, c_{n-3})$ where S_2 is irreducible, we would like to speak of the top edge in S_2 and the leftmost edge of S_1. Because of the cyclic nature of these ideas, we need to take a bit more care in our definitions.

Definition 4.8.5. Let $S_2 \subseteq \mathcal{U}$ be irreducible. Then there exists $\mu \in S_2$ such that $\mathrm{sh}(\mu; S_2)$ contains $\mathrm{sh}(\mu'; S_2)$ for all $\mu' \in S_2$, which is either unique or not unique. If μ is unique, then we call it the *top edge of* S_2. If μ is not unique, it follows from the definition of local shadows that $\mathrm{sh}(\mu; S_2) = \mathcal{R}$ for all such μ's. Let μ_{i_o} be the unique edge such that $\mathrm{sh}(\mu_{i_o}; S_2) = \mathcal{R}$, but the local shadow of the vertical edge in S_2 immediately below it is not $\mathcal{R}$. Define the following order on vertical edges in $\mathcal{U}$: $\mu_{i_o} < \mu_{i_o+1} < \cdots < \mu_{i_o+c_{n-3}-1}$. We say that the *top edge of* S_2 is the largest μ under this ordering such that $\mathrm{sh}(\mu; S_2) = \mathcal{R}$.

Definition 4.8.6. Let $(S_1, S_2) \in \mathrm{Pairs}(c_{n-2}, c_{n-3})$. Let S_2 be irreducible and let ρ_{i_o} be the horizontal edge which is immediately before the top edge of S_2. This edge defines the following order on horizontal edges in $\mathcal{R}$: $\mu_{i_o+1} < \mu_{i_o+2} < \cdots < \mu_{i_o+c_{n-2}} = \mu_{i_o}$. Let the leftmost edge of S_1 be the smallest edge in S_2 under this ordering.

With these concepts, we can now state a technical lemma, which is key in proving that certain edges are compatible.

Lemma 4.8.7. Let $(S_1, S_2) \in \widetilde{\mathcal{C}}_n$ where S_2 is irreducible. Let (ρ, μ) be such that $\mu \in S_2$ is the top edge of S_2 and $\rho \in \mathrm{sh}(\mu; S_2)$. Let $E = (x_1, y_1)$ be the left endpoint of ρ, (x_2, y_2) be the lower endpoint of μ, G be the upper endpoint of μ. Let $h = y_2 - y_1$, understood modulo c_{n-3} as a positive integer. Then if $h \geq r|(EG)_1 \cap S_1|$, there exists $F \in (EG)^\circ$ such that $r|(EF)_1 \cap S_1| = |(EF)_2|$.

Proof. Consider the function $f : (EG)^\circ \to \mathbb{Z}$ defined by $f(F) = |(EF)_2| - r|(EF)_1 \cap S_1|$. Notice that as we traverse the segment EG from E to G, whenever f increases, it increases by 1. Evaluating f at the right endpoint of ρ yields a negative value, and the condition of the problem implies that $f(x_2, y_2) \geq 0$. Therefore, there exists $F \in (EG)^\circ$ such that $f(F) = 0$, as desired. □

Remark 4.8.8. We expect this lemma to be an if-and-only-if statement.

For the remainder of this section, we will work to understand how colors of paths relate to the corresponding compatibility conditions, and then prove two special cases of Conjecture 4.

Definition 4.8.9. Let $r \geq 2$. Given a positive integer n, its *r-greedy decomposition* $n = \sum_{m\geq 1} a_m c_m$, or simply *greedy decomposition*, is defined recursively as follows: there exists a largest m such that for some $1 \leq a_m \leq r-1$ and $a_m c_m \leq n < \min((a_m+1)c_m, c_{m+1})$; then the greedy decomposition of n is $a_m c_m$ plus the greedy decomposition of $n - a_m c_m$.

In other words, to greedily decompose n, we simply use as many of the largest c_m's as possible. In light of Lemma 4.4.5, the greedy decomposition can be applied to either coordinate of a vertex to quickly locate it on the maximal Dyck path $\mathcal{D}_{c_{m+1}\times c_m}$.

Example 4.8.10. The 3-greedy decomposition of $n = 14$ is $8 + 2 \cdot 3$. Since $w(55,21) = w(21,8)^2 w(13,5) = w(21,8)w(8,3)^2 w(5,2)w(13,5)$, the position of v_{14} in the maximal Dyck path $\mathcal{D}_{55\times 21}$ is within the second appearance of $w(21,8)$, and more precisely right after the two blocks of $w(8,3)$. See Figure 4.9.

The maximal Dyck path interpretation of the greedy decomposition and Lemma 4.4.5 together imply the following property of greedy decompositions.

Lemma 4.8.11. Let $n > 0$ and let its greedy decomposition be $n = \sum_m a_m c_m$. If $a_{k+1} = r-1$, then $a_k < r-1$.

When $i = 0$, $s_{0,j} \leq s$ for all j, which implies that $\alpha(0,j)$ is always blue. In the following theorem, we consider the remaining cases.

Lemma 4.8.12. Let $r \geq 2$, $i > 0$ and let $i = \sum_{m\geq 1} a_m c_m$ be the r-greedy decompositon of i. If m_o is the smallest integer $m \geq 1$ such that $a_m \neq 0$, let $C = c_{m_o+1}$ if $0 \leq a_{m_o+1} < r-1$ and let $C = c_{m_o+1} - c_{m_o}$ if $a_{m_o+1} = r-1$.

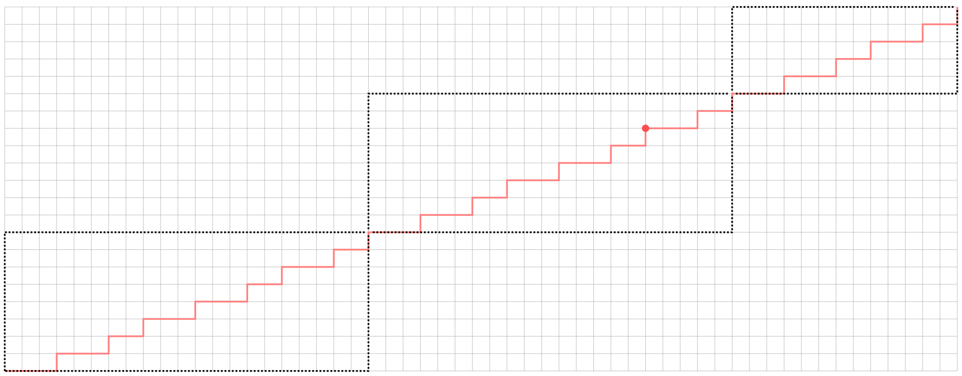


Figure 4.9 Illustration of $\mathcal{D}_{55,21}$ and the position of the vertex v_{14}

Then $j_o = i + C - a_{m_o} c_{m_o}$ is the smallest integer $t > i$ such that $s_{i,t} > s$, so that $\alpha(i,j) \in \mathcal{F}(\mathcal{D}_n)$ is a blue path if and only if $j < j_o$. If $j \geq j_o$, $\alpha(i,j)$ is red if $j_o - i = 1$. Otherwise, $\alpha(i,j)$ is $(m_o + 1, a_{m_o})$-green if $C = c_{m_o+1}$, and $(m_o + 1, a_{m_o} + 1)$-green if $C = c_{m_o+1} - c_{m_o}$.

Proof. First we will show that whenever $t < j_o$, $s_{i,t} < s$. Since the sequence $\frac{c_m}{c_{m+1}}$ is increasing (see Proposition A.0.5), it suffices to show that $s_{i,t} \leq \frac{c_{m_o}}{c_{m_o+1}}$. We may write $t - i = \sum_{\alpha=1}^{k} a_{m_\alpha} c_{m_\alpha}$ such that $1 \leq a_{m_\alpha} \leq r - 1$, $m_k < m_{k-1} < \cdots < m_1 \leq m_o$. By Lemma 4.4.5,

$$\bar{w}(i,t) = w(c_{m_1}, c_{m_1+1})^{a_{m_1}} \cdots w(c_{m_k}, c_{m_k+1})^{a_{m_k}}.$$

Since $m_k < m_{k-1} < \cdots < m_1 \leq m_o$, indeed

$$s_{i,j} \leq \frac{c_{m_1}}{c_{m_1+1}} \leq \frac{c_{m_o}}{c_{m_o+1}}$$

as desired.

Now suppose that $j \geq j_o$. If $a_{m_o+1} < r-1$, then by Lemma 4.4.5, we may view $\alpha(i, j_o)$ as a tail of the maximal Dyck path in $R_{c_{w_o+2} \times c_{w_o+1}}$, so that the slope s_{i,j_o} may be computed as follows:

$$s_{i,j_o} = \frac{c_{m_o+1} - a_{m_o} c_{m_o}}{c_{m_o+2} - a_{m_o} c_{m_o+1}}.$$

If $a_{m_o+1} = r - 1$, then we may view $\alpha(i, j_o)$ as a tail of the maximal Dyck path in $R_{(c_{m_o+2} - c_{m_o+1}) \times (c_{m_o+1} - c_{m_o})}$, and

$$s_{i,j_o} = \frac{c_{m_o+1} - c_{m_o} - a_{m_o} c_{m_o}}{c_{m_o+2} - c_{m_o+1} - a_{m_o} c_{m_o+1}}.$$

Note that by Lemma 4.8.11, in this case, $1 + a_{m_o} \leq r - 1$. Therefore, in both cases, we may apply Proposition A.0.6 to get that $s_{i,j_o} > s$. This shows the first part of the theorem.

The first part of the theorem shows that for $j \geq j_o$, $\alpha(i, j)$ and $\alpha(i, j_o)$ have the same color. So to finish the proof, it suffices show that $\alpha(i, j_o)$ is red if and only if $j_o - i = 1$. Recall that $\alpha(i, j_o)$ is green if we can write

$$j_o - i = c_m - wc_{m-1}$$

where $3 \leq m \leq n - 2$ and $1 \leq w < r - 1$. But

$$c_m - wc_{m-1} \geq c_m - (r-2)c_{m-1} = c_{m-1} + (c_{m-1} - c_{m-2}) > 1.$$

Conversely, suppose that $j_o - i \neq 1$. We can always write

$$j_o - i = c_{m_o+1} - wc_{m_o} \tag{4.8}$$

where $1 \leq w \leq r - 1$ and $m_o \geq 2$. If $w < r - 1$, then Equation 4.8 shows that $\alpha(i, j_o)$ is green. If $w = r - 1$, it follows from Equation 4.8 that

$$j_o - i = c_{m_o+1} - (r-1)c_{m_o} = c_{m_o} - c_{m_o-1}.$$

Since $j_o - i \neq 1$, we must have $m_o \neq 2$. So $m_o \geq 3$ and $\alpha(i, j_o)$ is green. □

We also need a quick lemma which tells us about quantities related to traversing the maximal Dyck path backwards, as remote shadows do.

Lemma 4.8.13. Let $(a, b) = (c_{n+1}, c_n)$ or $(c_{n+1} - c_n, c_n - c_{n-1})$. For $0 \leq i < a$, let the highest point on the maximal Dyck path $\mathcal{D}_{a\times b}$ with horizontal coordinate $a - i$ be $(a - i, b - j)$. Let the greedy decomposition of i be $\sum_{m\geq 1} a_m c_m$. Then $j = \sum_{m\geq 1} a_m c_{m-1} + 1$.

Proof. By Lemma 4.4.5, if (i, j) is the highest point on the maximal Dyck path $\mathcal{D}_{a\times b}$ with horizontal coordinate i, and $i = \sum_{m\geq 1} a_m c_m$, then $j = \sum_{m\geq 1} a_m c_{m-1}$.

The lemma then follows from the above observation and Proposition 4.4.6. □

Theorem 4.8.14. Given i, let j_o be as defined in Lemma 4.8.12. Conjecture 4 is true if $S_2 = \{\mu_i, \ldots, \mu_t\}$, where $t \leq j_o$.

Proof. Let us first consider the case where $\alpha(i,t)$ is blue, i.e. when $t < j_o$. Since $\mathcal{F}(\mathcal{D}_n)$ places no restrictions on the inclusion of additional edges, we have that $\widetilde{\Phi}^{-1}(S_1,S_2) \in \mathcal{F}(\mathcal{D}_n)$ for any choice of $S_1 \subseteq \operatorname{rsh}(S_2)$, so it suffices to show that when $S_2 = \{\mu_i,\ldots,\mu_t\}$ where $t < j_o$, and when $S_1 \subseteq \operatorname{rsh}(S_2)$, the pair (S_1,S_2) is always compatible. We proceed by strong induction. In the base case where $t = i+1$, since $\operatorname{rsh}(S_2) = \emptyset$, the claim is trivially true. Next, consider a generic $\alpha(i,t)$ where $t < j_o$. Since $\alpha(i,k)$ is blue for all $i < k \le t$, it suffices to consider the compatibility of pairs (ρ,μ_t), where $\rho \in S_1$. Let ρ be the leftmost edge of $\operatorname{rsh}(\mu_t;S_2) = \operatorname{rsh}(S_2)$. For every $\rho' \in S_1$, there exists some smallest $i < k \le t$ such that $\rho' \in \operatorname{rsh}(\mu_k;S_2)$. By induction, we know that ρ and μ_t are compatible, and the choice of a point which demonstrates their compatibility will also demonstrate the compatibility of ρ' and μ_t. Therefore, it suffices to show that the edges (ρ,μ_t) are compatible.

Let $E = (x_1,y_1)$ be the left endpoint of ρ, $F = v_i$, and $G = (x_2,y_2{+}1) = v_t$. Let $h_1 \equiv_{c_{n-3}} y_2 - i$, $h_2 \equiv_{c_{n-3}} i - y_1$.

If $|(FG)_1| = \sum_{m\ge1} a_m c_m$,

$$\begin{aligned} h_1 + 1 = |(FG)_2| &= \sum_{m\ge1} a_m c_{m-1}; \\ |\operatorname{rsh}(S_2)| = r|(FG)_2| - |(FG)_1| &= \sum_{m\ge1} a_m c_{m-2}; \\ h_2 &= \sum_{m\ge1} a_m c_{m-3} + 1, \end{aligned}$$

where the first two equations are due to Lemma 4.4.5, and the last is due to Lemma 4.8.13.

Recall that $c_{m-3} + c_{m-1} = rc_{m-2}$. By Lemma 4.8.7, since $h_1 + h_2 = r|\operatorname{rsh}(S_2)| \ge r|S_1|$, we conclude that (ρ,μ_t) are compatible.

Now consider the case where $t = j_o$, in which case $\alpha(i,t)$ is red or green. Since $t = j_o$ is the first $k > i$ such that $\alpha(i,k)$ is not blue, we know that (ρ,μ_k) is compatible for all $i \le k < t$ and $\rho \in S_1$. Therefore, for the same reason as before, it suffices to show that the edges (ρ,μ_t) are compatible, where ρ is the leftmost edge of $\operatorname{rsh}(\mu_t;S_2) = \operatorname{rsh}(S_2)$. Define E,F,G,y_1,y_2,h_1,h_2 in the same manner as in the previous case. Since $t = j_o$, we may write $h_1+1 = t-i = c_m - wc_{m-1}$ where $1 \le w \le r-1$ and $m > 1$. Then by Lemma 4.4.5, we have the following,

$$\begin{aligned} |\operatorname{rsh}(S_2)| = r|(FG)_2| - |(FG)_1| &= c_{m-1} - wc_{m-2}; \\ h_2 &= c_{m-2} - wc_{m-3}. \end{aligned}$$

Therefore, $h_1 + h_2 = c_m - wc_{m-1} + c_{m-2} - wc_{m-3} - 1 = r|\operatorname{rsh}(S_2)| - 1$. Since $\operatorname{rsh}(S_2)$ in this case clearly consists of consecutive horizontal edges, the converse of Lemma 4.8.7 is also true. Therefore, the edges (ρ, μ_t) are compatible if and only if $h_1 + h_2 \geq r|S_1|$, which occurs if and only if $|S_1| < |\operatorname{rsh}(S_2)|$. This corresponds exactly to the requirement that for $\beta \in \mathcal{F}(\mathcal{D}_n)$, if $\alpha(i, j) \in \beta$ is red, the immediately preceding edge of $\alpha(i, j)$ must be included in β, and if $\alpha(i, j) \in \beta$ is (m, w)-green, then one of the previous $c_{m-1} - wc_{m-2}$ edges must be included in β.

$\square$

Chapter 5

F-Polynomial Limits in $\widetilde{A_{n,1}}$

In this chapter, we discuss limits of certain ratios of F-polynomials in the cluster algebra $\widetilde{A_{n,1}}$. In Section 5.1, we will introduce $\widetilde{A_{n,1}}$, establish some useful notations and discuss the connections between $\widetilde{A_{n,1}}$ and triangulations of a certain surface. In Section 5.2, we introduce (generalized) continued fractions, which turn out to be a surprisingly suitable approach for understanding the limits that we are concerned with. In Section 5.3, we will provide an exposition of a variety of approaches to a particular F-polynomial limit in the Kronecker case, which acts as the starting point for our work. In Section 5.4, we will state and prove our results about analogous limits of F-polynomials in the $\widetilde{A_{n,1}}$ cluster algebra.

5.1 The Cluster Algebra $\widetilde{A_{n,1}}$

Recall from Chapter 1 that the cluster algebra $\widetilde{A_{n,1}}$ is the skew-symmetric cluster algebra defined by the quiver $Q_{n,1}$.

$$\begin{tikzcd} 1 \arrow[r] \arrow[rrr, bend left] & 2 \arrow[r] & \cdots \arrow[r] & n+1 \end{tikzcd}$$

Figure 5.1 The quiver $Q_{n,1}$

One may check that the Cartan companion (see Section 2.4) of the adjacency matrix of $Q_{n,1}$ has corank equal to 1, which shows that $\widetilde{A_{n,1}}$ is an affine cluster algebra. Compared to the r-Kronecker, which is of indefinite type when $r \geq 3$, the cluster algebra $\widetilde{A_{n,1}}$ is an affine generalization of the Kronecker cluster algebra. See also Remark 5.3.1 for an explanation of why we do not expect our specific line of investigation to work out with the r-Kronecker.

The cluster algebra $\widetilde{A_{n,1}}$ is of *surface type*. For a rigorous and general introduction to cluster algebras of surface type, we point the reader to Section 2 of Fomin et al. (2008). We will proceed to introduce what being of surface type means for $\widetilde{A_{n,1}}$ in a more hands-on manner. Let $T_{n,1}$ be the annulus with two boundary components, n marked points on the outer boundary, and 1 marked point on the inner boundary.

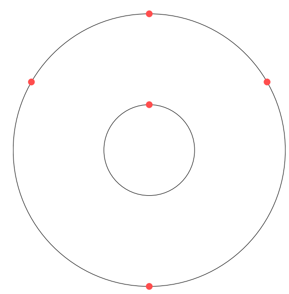


Figure 5.2 The annulus $T_{4,1}$

Then cluster variables of $\widetilde{A_{n,1}}$ are in bijection with (isotopy classes[1] of) arcs that connect two marked points on $T_{n,1}$, and clusters of $\widetilde{A_{n,1}}$ are in bijection with (ideal) triangulations[2] of $T_{n,1}$. In the case of $T_{n,1}$, by convention, the triangulation given in Figure 5.3 is taken to be the *initial triangulation*, namely, the triangulation that corresponds to the initial seed.

Given a triangulation, there is a procedure for obtaining a quiver, whose adjacency matrix is the exchange matrix of the corresponding cluster. We consider every marked point, which is the endpoint of some possibly empty collection of arcs. For each such marked point, we draw counterclockwise edges between adjacent arcs. This creates a quiver whose vertices are the arcs of the triangulation and whose edges are determined by the arrows we have drawn. Figure 5.4 shows how we obtain the quiver $Q_{3,1}$ from the initial triangulation of $T_{3,1}$. The labeling of an edge denotes the initial cluster

[1]Two arcs are in the same isotopy class if we can deform one into the other without touching the annulus.

[2]Intuitively, one could understand ideal triangulations as triangulations where we replace straight edges with arcs. Just like how triangulations of a polygon consist of maximal non-crossing collections of diagonals, ideal triangulations consist of maximal non-crossing collections of these arcs, where we say two arcs don't cross if there are representatives of their isotopy classes which don't cross.

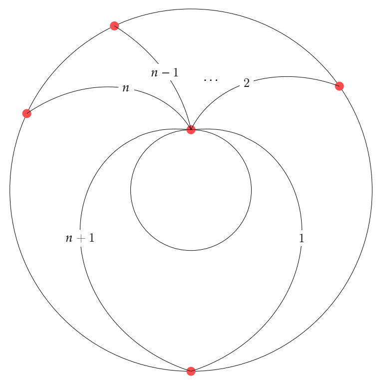
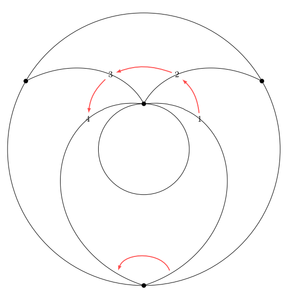


Figure 5.3 The initial triangulation of $T_{n,1}$

variable to which the edge corresponds. Mutations can also be understood from the perspective of arcs and triangulations: every arc borders two ideal triangles in a triangulation, which together forms a quadrilateral in which the arc is a diagonal; a mutation at a cluster variable corresponds to replacing the corresponding arc by the other diagonal of the quadrilateral.



$$\begin{tikzcd} 1 \arrow[r] \arrow[rrr, bend left] & 2 \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd}$$

Figure 5.4 Obtaining the initial quiver $Q_{3,1}$ from the initial triangulation of $T_{3,1}$

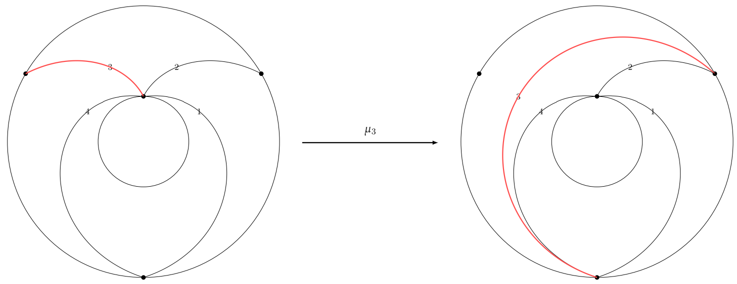


Figure 5.5 Mutation at an arc

Let μ_+ be the infinite sequence of mutations given by repeated applications of $\mu_1\mu_2\cdots\mu_{n+1}$, and let μ_- be the infinite sequence of mutations given by repeated applications of $\mu_{n+1}\mu_n\cdots\mu_1$. For $m \geq 1$, let $\mu_+(m)$ and $\mu_-(m)$ denote the sequence of the first m mutations of μ_+ and μ_-. Let x_{n+m} denote the new cluster variable from applying the last mutation in the sequence $\mu_+(m)$ and let x_{1-m} denote the new cluster variable from applying the last mutation in the sequence $\mu_-(m)$. We will use $\mathbf{g}_m, F_m$ to denote the $\mathbf{g}$-vectors and F-polynomials associated to the cluster variable x_m. Similar to the rank-two case, any x_m appears in $n+1$ different clusters along the μ_+ and μ_- sequence, and has a $\mathbf{c}$-vector associated to it at each of these clusters. We let $\mathbf{c}_m$ denote the $\mathbf{c}$-vector of x_m at the cluster along the μ_+ or μ_- sequences which is closest to the initial cluster.

The mutation sequences μ_+ and μ_- have the properties of being a *source mutation sequence* and a *sink mutation sequence* respectively. A *source (resp. sink) mutation sequence* is a sequence of mutations which at each step, is at a source (resp. sink) of the underlying acyclic quiver. One may check that μ_+ and μ_- are the unique source and sink sequences that start at the initial seed. For instance, the vertex 1 is the unique source of the quiver $Q_{n,1}$, and the vertex $n+1$ is the unique sink of the quiver $Q_{n,1}$. Mutations along the source and sink sequences are always exchanging a bridging arc by another bridging arc. Using the surface interpretation, one easily checks that every cluster variable that corresponds to a bridging arc appears as x_m for some nonzero $m \in \mathbb{Z}$.

5.2 Some Background on Continued Fractions

Traditionally, continued fractions provide a correspondence between $\mathbb{R}_{\geq 1}$ and positive integer sequences. Given a possibly infinite sequence of positive integers $a_0, a_1, a_2, \ldots$, the continued fraction $[a_0; a_1, a_2, \ldots]$ is a real number defined by

$$[a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}.$$

Conventionally, we write a semicolon after a_0 to distinguish it form the rest of the a_i's because it is the integer part of the real number defined by this continued fraction. Conversely, given a real number r, a recursive process allows us to express it in the form $[a_0; a_1, a_2, \ldots]$. One starts with $r_0 = r$, and let $a_0 = \lfloor r_0 \rfloor$, which leaves us with $r_1 = \frac{1}{r_0 - a_0}$. At each stage, $a_k = \lfloor r_k \rfloor$, and $r_{k+1} = \frac{1}{r_k - a_k}$. Visually, we have that for each $k \geq 0$,

$$r = a_0 + \cfrac{1}{\cdots + \cfrac{\cdots}{a_k + \cfrac{1}{r_k}}}.$$

Under this correspondence, rational numbers correspond to finite continued fractions and irrational numbers correspond to infinite continued fractions.

Example 5.2.1. A continued fraction evaluates to a quadratic irrational if and only if it is periodic[3]. As an example, we evaluate two continued fractions related to our work. We claim that for $n \geq 1$ and $1 < k \leq n$,

$$[2; \overline{n, 1}] = \frac{3n + \sqrt{n^2 + 4n}}{2n}, \tag{5.1}$$

$$[1; k-1, \overline{1, n}] = \frac{2k(n-k+1) + n + \sqrt{n^2 + 4n}}{2kn - 2(k-1)^2}. \tag{5.2}$$

Let $z = [n; \overline{1, n}]$. Then by the periodicity of this continued fraction,

$$z = n + \cfrac{1}{1 + \cfrac{1}{z}}.$$

Solving for z, we obtain

$$z = \frac{n \pm \sqrt{n^2 + 4n}}{2}.$$

[3]See Theorem 176 and 177 of Hardy and Wright (2008).

Since z is evidently positive and $n - \sqrt{n^2+4n} < 0$, we must have

$$z = \frac{n + \sqrt{n^2+4n}}{2}.$$

It follows that

$$[2; \overline{n,1}] = 2 + \frac{1}{z} = \frac{3n + \sqrt{n^2+4n}}{2n}.$$

To evaluate the left-hand side of Equation 5.2, let $y = [1; \overline{n,1}]$. Then

$$y = 1 + \frac{1}{z} = \frac{n + \sqrt{n^2+4n}}{2n}.$$

Now

$$[1; k-1, \overline{1,n}] = 1 + \cfrac{1}{k - 1 + \cfrac{1}{y}},$$

which, after some algebra, simplifies to the right-hand side of Equation 5.2.

Given an infinite continued fraction $[a_0; a_1, a_2, \ldots]$, the rational number $c_n = [a_0; a_1, \ldots, a_n]$ is called a *continuant*. The sequence $\{c_n\}_{n\in\mathbb{Z}}$ of continuants is a sequence of rational numbers that approaches the exact continued fraction. It is well-known classically that if we let $c_n = \frac{h_n}{k_n}$, that h_n and k_n can be found recursively:

$$h_{-2} = 0,\ h_{-1} = 1,\ h_n = a_n h_{n-1} + h_{n-2}; \tag{5.3}$$

$$k_{-2} = 1,\ k_{-1} = 0,\ k_n = a_n k_{n-1} + k_{n-2}. \tag{5.4}$$

Moreover, $h_n k_{n-1} - h_{n-1} k_n = (-1)^{n+1}$, which implies that

$$c_n - c_{n-1} = \frac{h_n}{k_n} - \frac{h_{n-1}}{k_{n-1}} = \frac{(-1)^{n+1}}{k_n k_{n-1}}.$$

Example 5.2.2. Consider the real number $\frac{3+\sqrt{5}}{2} = [2; \overline{1}]$, which is the Golden ratio plus one. It is well-known that the golden ratio is approximated by ratios of adjacent Fibonacci numbers. We calculate that

$$h_0 = 2;\ h_1 = 3,\ h_2 = 5,\ \ldots$$

$$k_0 = 1;\ k_1 = 1,\ k_2 = 2,\ \ldots$$

which one might notice are the Fibonacci numbers. This suggests that the sequence of continuants of $\frac{3+\sqrt{5}}{2}$ is

$$\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{13}{5}, \dots$$

the ratio of every other Fibonacci, which agrees with our understanding of the golden ratio.

In our application, we will need to slightly generalize the classical continued fraction. Let R be $\mathbb{Z}[y_1, \dots, y_n]$. A *generalized continued fraction over* R, notated $[[a_0, a_1, a_2, \dots], [b_1, b_2, \dots]]$, where $a_i, b_i \in R$, is notated as

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cdots}},$$

which is an element of $\mathbb{Q}[[y_1, \dots, y_n]]$. We understand b_0 to be 1. For $n \geq 0$, we similarly define continuants to be

$$c_n = [[a_0; a_1, \dots, a_n], [b_1, \dots, b_n]],$$

which are rational functions in $y_1, \dots, y_n$ but could be understood as power series as well by Taylor-expanding the denominator at the origin.

We shall prove a result about these continuants analogous to the classical scenario.

Theorem 5.2.3. Let c_n be the n-th continuant of a generalized infinite continued fraction $[[a_0, a_1, a_2, \dots], [b_1, b_2, \dots]]$. Then $c_n = \frac{h_n}{k_n}$, where

$$h_{-2} = 0, h_{-1} = 1, h_n = a_n h_{n-1} + b_n h_{n-2}; \tag{5.5}$$
$$k_{-2} = 1, k_{-1} = 0, k_n = a_n k_{n-1} + b_n k_{n-2}. \tag{5.6}$$

Moreover, $h_n k_{n-1} - h_{n-1} k_n = (-1)^{n+1} b_1 b_2 \cdots b_n$, which implies that

$$c_n - c_{n-1} = \frac{h_n}{k_n} - \frac{h_{n-1}}{k_{n-1}} = \frac{(-1)^{n+1} b_1 b_2 \cdots b_n}{k_n k_{n-1}}.$$

Proof. We prove that $c_n = \frac{h_n}{k_n}$ by induction on n. When $n = 0$,

$$h_0 = a_0 h_{-1} + b_0 h_{-2} = a_0, \quad k_0 = a_0 k_{-1} + b_0 k_{-2} = 1,$$

which gives us $\frac{h_0}{k_0} = a_0 = [[a_0],[]] = c_0$, as expected. When $n = 1$,

$$h_1 = a_1 h_0 + b_1 h_{-2} = a_1 a_0 + b_1, \quad k_1 = a_1 k_{-1} + b_1 k_{-2} = a_1,$$

so indeed $\frac{h_1}{k_1} = \frac{a_1 a_0 + b_1}{a_1} = [[a_0; a_1],[b_1]] = c_1$. Now suppose that the claim is true for $n = m-2, m-1$. Then

$$\begin{aligned} c_m &= [[a_0; a_1, \ldots, a_{m-2}, a_{m-1} + \frac{b_m}{a_m}], [b_1, \ldots, b_{m-1}]] \\ &= \frac{(a_{m-1} + \frac{b_m}{a_m})h_{m-2} + b_{m-1}h_{m-3}}{(a_{m-1} + \frac{b_m}{a_m})k_{m-2} + b_{m-1}k_{m-3}} \\ &= \frac{h_{m-1} + \frac{b_m}{a_m}h_{m-2}}{k_{m-1} + \frac{b_m}{a_m}k_{m-2}} \\ &= \frac{a_m h_{m-1} + b_m h_{m-2}}{a_m k_{m-1} + b_m k_{m-2}} \\ &= \frac{h_m}{k_m}. \end{aligned}$$

We will now show the claim that $h_n k_{n-1} - h_{n-1} k_n = (-1)^{n+1} b_1 b_2 \cdots b_n$. We may verify this for $n = 0$:

$$h_0 k_{-1} - h_{-1} k_0 = a_0 \cdot 0 - 1 \cdot 1 = -1.$$

Now suppose that the claim is true for $n = m-1$. Then

$$\begin{aligned} h_m k_{m-1} - h_{m-1} k_m &= (a_m h_{m-1} + b_m h_{m-2})k_{m-1} - h_{m-1}(a_m k_{m-1} + b_m k_{m-2}) \\ &= b_m h_{m-2} k_{m-1} - b_m k_{m-2} h_{m-1} \\ &= -b_m (-1)^m b_1 b_2 \cdots b_{m-1} \\ &= (-1)^{m+1} b_1 b_2 \cdots b_m \end{aligned}$$

as desired. □

Corollary 5.2.4. Given the same setup as Theorem 5.2.3, for $n \geq 0$,

$$c_{2n+2} - c_{2n} = \frac{a_{2n+2} b_1 b_2 \cdots b_{2n+1}}{k_{2n+2} k_{2n}}.$$

Proof.

$$\begin{aligned}
c_{2n+2} - c_{2n} &= (c_{2n+2} - c_{2n+1}) + (c_{2n+1} - c_{2n}) \\
&= (-1)^{2n+2+1}\frac{b_1 b_2 \cdots b_{2n+2}}{k_{2n+2}k_{2n+1}} + (-1)^{2n+2}\frac{b_1 b_2 \cdots b_{2n+1}}{k_{2n+1}k_{2n}} \\
&= \frac{-b_{2n+2}k_{2n} + k_{2n+2}}{k_{2n+2}k_{2n+1}k_{2n}} b_1 b_2 \cdots b_{2n+1} \\
&= \frac{a_{2n+2}b_1 b_2 \cdots b_{2n+1}}{k_{2n+2}k_{2n}},
\end{aligned}$$

where the last equality is due to $k_{2n+2} = a_{2n+2}k_{2n+1} + b_{2n+2}k_{2n}$. ☐

5.3 An F-Polynomial Limit in the Kronecker Quiver

The starting point for this work is a result due to Canakci and Schiffler (2017) about the limit of ratios of certain cluster variables in the cluster algebra of the once-punctured torus, which specializes to the Kronecker quiver $\widetilde{A}_{1,1}$. This result is later rediscovered in Reading (2020b) in the form of F-polynomials.

Specifically, they considered the power series

$$\mathcal{N}(\widehat{y}_1, \widehat{y}_2) := \lim_{i\to\infty} \frac{F_{i+1}(\widehat{y}_1, \widehat{y}_2)}{F_i(\widehat{y}_1, \widehat{y}_2)} = \lim_{i\to-\infty} \frac{F_{i-1}(\widehat{y}_1, \widehat{y}_2)}{F_i(\widehat{y}_1, \widehat{y}_2)}.$$

Since all F-polynomials have a constant term 1, both of the ratios $\frac{F_{i+1}(\widehat{y}_1,\widehat{y}_2)}{F_i(\widehat{y}_1,\widehat{y}_2)}$ and $\frac{F_{i-1}(\widehat{y}_1,\widehat{y}_2)}{F_i(\widehat{y}_1,\widehat{y}_2)}$ can be understood as a power series by Taylor-expanding the denominator at $\widehat{y}_1 = \widehat{y}_2 = 0$. It is a not-so-obvious fact that the coefficients of these power series stabilize as we take $i \to \infty$ or $i \to -\infty$, so that we can say that a limit power series $\mathcal{N}$ exists. There are several perspectives we can take to try to understand this power series. We may describe the coefficients explicitly, which are parameterized by the possible exponents on $\widehat{y}_1, \widehat{y}_2$. We may also try to write this power series in a compacter form. This is analogous to how there is a generating function for the Narayana numbers which involves a square root. Reading (2020b) provides both of these perspectives on $\mathcal{N}$ as he proves that

$$\mathcal{N}(\widehat{y}_1, \widehat{y}_2) = 1 + \widehat{y}_1 \sum_{i,j\geq 0} (-1)^{i+j}\operatorname{Nar}(i,j)\widehat{y}_1^i \widehat{y}_2^j \tag{5.7}$$

$$= \frac{1}{2}\left(1 + \widehat{y}_1 + \widehat{y}_1\widehat{y}_2 + \sqrt{1 + 2\widehat{y}_1(1-\widehat{y}_2) + \widehat{y}_1^2(1+\widehat{y}_2)^2}\right), \tag{5.8}$$

where for $i, j \geq 0$, $\mathrm{Nar}(i, j)$ denotes the Narayana number

$$\mathrm{Nar}(i,j) = \begin{cases} 1 & \text{if } i = j = 0, \\ 0 & \text{if } ij = 0, \\ \frac{1}{i}\binom{i}{j}\binom{i}{j-1} & \text{if } i \geq 1 \text{ and } j \geq 1. \end{cases}$$

In Canakci and Schiffler (2017), they understood this power series as an infinite (generalized) continued fraction that involves only Laurent monomials in $\widehat{y}_1$ and $\widehat{y}_2$. The way that they determine the specific Laurent polynomials is related to the so-called snake graph associated to a cluster variable of surface type. Canakci and Schiffler (2017) states their result for cluster variables. The analogous result for F-polynomials would be the following:

$$\mathcal{N}(\widehat{y}_1, \widehat{y}_2) = [[1 + \widehat{y}_1, \overline{1, \widehat{y}_1}], [\overline{\widehat{y}_1\widehat{y}_2, 1}]].$$

Lastly, observe that the difference between the $\mathbf{g}$-vectors associated to F_{i+1} and F_i is constant:

$$\mathbf{g}_{i+1} - \mathbf{g}_i = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Let γ_∞ and $\gamma_{-\infty}$ be the paths labeled in Figure 5.6. Nathan Reading observes that the limit $\lim_{i\to\infty} \frac{F_{i+1}(\widehat{y}_1,\widehat{y}_2)}{F_i(\widehat{y}_1,\widehat{y}_2)}$ can be understood as the path-ordered product of $\mathbf{x}^{\mathbf{g}_{i+1}-\mathbf{g}_i} = x_1^{-1}x_2$ under the path γ_∞, and a similar statement holds for $\lim_{i\to-\infty} \frac{F_{i-1}(\widehat{y}_1,\widehat{y}_2)}{F_i(\widehat{y}_1,\widehat{y}_2)}$ and $\gamma_{-\infty}$. See Reading (2020b) for a more precise statement and explanation. This perspective provides a visual intuition for why the limits in the positive and negative directions are equal: the limits of the two path-ordered products emanate from the same limiting wall in the scattering diagram, and crossing the limiting wall acts trivially on $x_1^{-1}x_2$.

In Chapter 3, we showed the equivalence of path-ordered products with Gupta's formula. However, Gupta's formula provides yet another perspective. This computation was first done by Gregg Musiker, who noticed that since

$$F_{\ell+2}(\widehat{y}_1, \widehat{y}_2) = L_1^{\ell} L_2^{\ell-1} \cdots L_\ell,$$

we have

$$\frac{F_{\ell+3}(\widehat{y}_1, \widehat{y}_2)}{F_{\ell+2}(\widehat{y}_1, \widehat{y}_2)} = L_1 L_2 \cdots L_{\ell+1},$$

and so

$$\mathcal{N}(\widehat{y}_1, \widehat{y}_2) = \prod_{i=1}^{\infty} L_i.$$

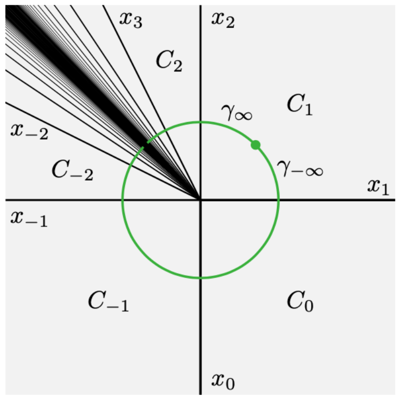


Figure 5.6 The path-ordered products $\mathfrak{p}_{\gamma_\infty}(x_1^{-1}x_2)$ and $\mathfrak{p}_{\gamma_{-\infty}}(x_1^{-1}x_2)$ are respectively the limits in the positive and negative directions in Equation 5.8; Figure 3 of Reading (2020b)

By Lemma 3.3.1,

$$L_1L_2\cdots L_\ell = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}^\ell} \prod_{j=1}^{\ell} \binom{1+\sum_{k=j+1}^{\ell} 2(j-k)m_k}{m_j} y_1^{\sum_{i=1}^{\ell} im_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}, \tag{5.9}$$

which implies that

$$\mathcal{N}(\widehat{y}_1,\widehat{y}_2) = \sum_{(m_1,m_2,\ldots)\in\mathbb{Z}_{\geq 0}^\infty} \prod_{j=1}^{\infty} \binom{1+\sum_{k=j+1}^{\infty} 2(j-k)m_k}{m_j} y_1^{\sum_{i=1}^{\infty} im_i} y_2^{\sum_{i=1}^{\infty}(i-1)m_i},$$

where the sum is over all nonnegative integer sequences that are only non-zero at finitely many indices. Because a sequence is only non-zero at finitely many indices, the infinite product $\prod_{j=1}^{\infty}\binom{1+\sum_{k=j+1}^{\infty}2(j-k)m_k}{m_j}$ is necessarily finite, and the sums $\sum_{i=1}^{\infty} im_i$ and $\sum_{i=1}^{\infty}(i-1)m_i$ are also finite.

Remark 5.3.1. Before we move into the next section and start discussing our results for $\widetilde{A_{n,1}}$, let us remark on why we did not look into limits of ratios of F-polynomials for rank-two cluster algebras. In this section, we saw that this question has a nice answer for the Kronecker quiver. To illustrate why there is not a similar answer for $\mathcal{A}(b,c)$ where $bc>4$, let us consider the case of $\mathcal{A}(3,3)$. If the ratio of F-polynomials tends to a certain limit, we expect that the specializations of these ratios under $y_1=y_2=1$, which would give a sequence of rational numbers, also converge. However, in the case of $\mathcal{A}(3,3)$, the first few $F_\ell(1,1)$ are

$$1, 1, 2, 9, 365, 5403014, 432130991537958813,$$
$$14935169284101525874491673463268414536523593057\ldots$$

The ratios of adjacent terms in this sequence clearly diverges.

However, there was actually no reason not to look into the analogous limit for $\mathcal{A}(1,4)$. This is a direction for future research, and we record some of our preliminary findings here.

Looking at the **g**-vectors of cluster variables in $\mathcal{A}(1,4)$, we find that they converge to the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, which corresponds to the slope of the limit ray in the scattering diagram for $\mathcal{A}(1,4)$. One can show that

$$2\mathbf{g}_{2m} - \mathbf{g}_{2m-1} = \mathbf{g}_{2m+1} - 2\mathbf{g}_{2m} = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

which suggests that the limit of the following sequence is the correct analogue of $\mathcal{N}$:

$$\frac{F_4^2}{F_3}, \frac{F_5}{F_4^2}, \frac{F_6^2}{F_5}, \frac{F_7}{F_6^2}, \ldots$$

Based on data, we found that as power series, this sequence appears to tend to a limit. Expressing each term of this sequence as a generalized continued fraction also suggested a pattern.

Let $\tilde{4} = 1 + y_1 + 2y_1y_2$, $\tilde{3} = 1 + 2y_2$.

n	Numerically	$[a_0, a_1, \ldots]$	$[b_1, b_2, \ldots]$
F_4^2/F_3	$[4; 1, 1]$	$[\tilde{4}, 1, y_1]$	$[y_1y_2^2, 1]$
F_5/F_4^2	$[4; 1, 1, 3, 1]$	$[\tilde{4}, y_1, 1, \tilde{3}, y_1]$	$[y_1^2y_2^2, 1, y_2^2, 1]$
F_6^2/F_5	$[4; 1, 3, 1, 1, 3, 1]$	$[\tilde{4}, y_1, \tilde{3}, y_1, 1, \tilde{3}, y_1]$	$[y_1^2y_2^2, 1, y_1y_2^2, 1, y_2^2, 1]$
F_7/F_6^2	$[4; 1, 3, 1, 1, 3, 1, 3, 1]$	$[\tilde{4}, y_1, \tilde{3}, y_1, 1, \tilde{3}, y_1, \tilde{3}, y_1]$	$[y_1^2y_2^2, 1, y_1y_2^2, 1, y_2^2, 1, y_1y_2^2, 1]$
F_8^2/F_7	$[4; 1, 3, 1, 3, 1, 1, 3, 1, 3, 1]$	$[\tilde{4}, y_1, \tilde{3}, y_1, \tilde{3}, y_1, 1, \tilde{3}, y_1, \tilde{3}, y_1]$	$[y_1^2y_2^2, 1, y_1y_2^2, 1, y_1y_2^2, 1, y_2^2, 1, y_1y_2^2, 1]$
F_9/F_8^2	$[4; 1, 3, 1, 3, 1,$ $1, 3, 1, 3, 1, 3, 1]$	$[\tilde{4}, y_1, \tilde{3}, y_1, \tilde{3}, y_1,$ $1, \tilde{3}, y_1, \tilde{3}, y_1, \tilde{3}, y_1]$	$[y_1^2y_2^2, 1, y_1y_2^2, 1, y_1y_2^2,$ $1, y_2^2, 1, y_1y_2^2, 1, y_1y_2^2, 1]$
F_{10}^2/F_9	$[4; 1, 3, 1, 3, 1, 3, 1,$ $1, 3, 1, 3, 1, 3, 1]$	$[\tilde{4}, y_1, \tilde{3}, y_1, \tilde{3}, y_1, \tilde{3}, y_1,$ $1, \tilde{3}, y_1, \tilde{3}, y_1, \tilde{3}, y_1]$	$[y_1^2y_2^2, 1, y_1y_2^2, 1, y_1y_2^2, 1, y_1y_2^2,$ $1, y_2^2, 1, y_1y_2^2, 1, y_1y_2^2, 1]$

Table 5.1 Continued Fraction Expansions of Ratios in $\mathcal{A}(1,4)$

We expect that in the limit, the red part takes over: the numerical limit should be $[4;\overline{1,3}] = \frac{1}{2}(5+\sqrt{21})$, and the power series should tend to

$$\begin{aligned}&[[\tilde{4},\overline{y_1,\tilde{3}}],[y_1^2y_2^2,\overline{1,y_1y_2^2}]]\\&=\frac{1}{2}\left(1+y_1+2y_1y_2+y_1y_2^2+\sqrt{(1+y_1+2y_1y_2-y_1y_2^2)^2+4y_1^2y_2^2(1+2y_2)}\right).\end{aligned}$$

We checked using Sage that this square root expression correctly predicts the coefficients of those $y_1^iy_2^j$ in $\frac{F_{17}}{F_{16}^2}$ with $i+j<20$.

5.4 F-Polynomial Limits in $\widetilde{A}_{n,1}$

5.4.1 The $\mathcal{N}_{k,n}$ Series and Continued Fractions

For $n \geq 1$ and $1 \leq k \leq n$, let

$$\mathcal{N}_{k,n}(y_1,\ldots,y_{n+1}) = \lim_{m\to\infty}\frac{F_{mn+k+1}(y_1,\ldots,y_{n+1})}{F_{mn+k}(y_1,\ldots,y_{n+1})}.$$

When $k = n = 1$, this specializes to the power series $\mathcal{N}$ discussed in Section 5.3. Since the following limits agree:

$$\lim_{m\to\infty}\frac{F_{mn+k+1}(y_1,\ldots,y_{n+1})}{F_{mn+k}(y_1,\ldots,y_{n+1})} = \lim_{m\to-\infty}\frac{F_{mn+k+1}(y_1,\ldots,y_{n+1})}{F_{mn+k}(y_1,\ldots,y_{n+1})},$$

which can be shown by writing these limits as path-ordered products and using the consistency of the scattering diagram for $\mathcal{A}_{n,1}$ (see Reading (2020b)), we restrict our attention to the positive direction of this limit in the following discussion.

Similarly to how we studied $\mathcal{N}$ in Section 5.3, we take several different perspectives to characterize $\mathcal{N}_{k,n}$. In this section, we state how $\mathcal{N}_{k,n}$ can be expressed in a closed form and as two different continued fractions. We will prove the equivalence of the closed form and the continued fraction expansions, and relay the proof of the theorems themselves to the next section. We shall need the following shorthand for polynomials that show up frequently in our expressions.

Definition 5.4.1. Let

$$\begin{aligned}P_+(y_1,\ldots,y_n) &= 1+y_1+\cdots+y_1y_2\cdots y_{n-1}+y_1y_2\cdots y_n,\\ P_-(y_1,\ldots,y_n) &= 1+y_1+\cdots+y_1y_2\cdots y_{n-1}-y_1y_2\cdots y_n.\end{aligned}$$

We fix the convention that $P_\pm(y_p,\ldots,y_q) = 1$ if $p = q+1$ and $P_\pm(y_p,\ldots,y_q) = 0$ if $p = q+2$.

Theorem 5.4.2. Let

$$\begin{aligned} B_{k,n} &= y_{k+1}(P_+(y_{k+2},\ldots,y_n) + y_1 P_+(y_2,\ldots,y_{k-1})P_+(y_{k+1},\ldots,y_{n+1})), \\ A_{k,n} &= B_{k,n} + P_+(y_1,\ldots,y_{k-1}), \\ \Delta_n &= P_-(y_1,\ldots,y_{n+1})^2 + 4y_1^2 y_2 \cdots y_{n+1} P_+(y_2,\cdots,y_n). \end{aligned}$$

Then for $1 \le k \le n$,

$$\mathcal{N}_{k,n}(y_1,\ldots,y_{n+1}) = \frac{P_+(y_1,\ldots,y_{n+1}) + 2B_{k,n} + \sqrt{\Delta_n}}{2A_{k,n}}. \tag{5.10}$$

Remark 5.4.3. The terms A_k and B_k are equivalently

$$\begin{aligned} A_{k,n} &= P_+(y_1,\ldots,y_{k-1})P_+(y_{k+1},\ldots,y_{n+1}) - y_{k+1}y_{k+2}\ldots y_{n+1}, \\ B_{k,n} &= A_{k,n} - P_+(y_1,\ldots,y_{k-1}). \end{aligned}$$

This gives a simpler way of computing $B_{k,n}$, but we chose to present Theorem 5.4.2 this way because it makes the coefficients of terms in $B_{k,n}$ evidently positive.

Example 5.4.4. If we let $k = n = 1$, then

$$A_{1,1} = 1, B_{1,1} = 0, \Delta_1 = P_-(y_1, y_2)^2 + 4y_1^2 y_2,$$

so Theorem 5.4.2 says that

$$\mathcal{N} = \mathcal{N}_{1,1} = \frac{1}{2}\left(1 + y_1 + y_1 y_2 + \sqrt{(1 + y_1 - y_1 y_2)^2 + 4y_1^2 y_2}\right),$$

which agrees with Equation 5.8. Beyond the $n = 1$ case, we provide the following table to give some intuition for the terms in Equation 5.10.

n	k	$A_{k,n}$	$B_{k,n}$
2	1	$1 + y_2$	y_2
2	2	$1 + y_1 + y_1 y_3$	$y_1 y_3$
3	1	$1 + y_2(1 + y_3)$	$y_2(1 + y_3)$
3	2	$1 + y_1 + y_3(1 + y_1 + y_1 y_4)$	$y_3(1 + y_1 + y_1 y_4)$
3	3	$1 + y_1 + y_1 y_2 + y_1 y_4(1 + y_2)$	$y_1 y_4(1 + y_2)$
n	1	$1 + y_2 P_+(y_3,\ldots,y_n)$	$y_2 P_+(y_3,\ldots,y_n)$
n	n	$P_+(y_1,\ldots,y_{n-1}) + y_1 y_{n+1} P_+(y_2,\ldots,y_{n-1})$	$y_1 y_{n+1} P_+(y_2,\ldots,y_{n-1})$

Table 5.2 Some Examples of $A_{k,n}$ and $B_{k,n}$

We are inspired by the Kronecker case to write the power series $\mathcal{N}_{k,n}$ as generalized infinite continued fractions. Under the specialization $y_i \mapsto 1$,

$$A_{k,n} \mapsto k(n+2-k)-1,\ B_{k,n} \mapsto k(n+1-k)-1,\ \Delta_n \mapsto n^2+4n,$$

so by Theorem 5.4.2,

$$\mathcal{N}_{k,n}\Big|_{y_i=1} = \frac{2kn-2k^2+2k+n+\sqrt{n^2+4n}}{2kn-2k^2+4k-2} = \frac{2k(n-k+1)+n+\sqrt{n^2+4n}}{2kn-2(k-1)^2}.$$

Therefore by Equation 5.1 and Equation 5.1, we expect a continued fraction formula for $\mathcal{N}_{k,n}$ to specialize as follows:

$$\mathcal{N}_{k,n}\Big|_{y_i=1} = \begin{cases} [2;\overline{n,1}], & \text{if } k=1, \\ [1;k-1,\overline{1,n}], & \text{if } 1<k\le n. \end{cases}$$

We provide two such continued fraction expansions.

Theorem 5.4.5. For $1 \le k \le n$, let

$$\alpha_{k,n} = \begin{cases} \frac{P_-(y_1,\ldots,y_{n+1})}{y_1^2 y_2 \cdots y_{n+1}}, & \text{if } k=1, \\ \frac{P_+(y_1,\ldots,y_{n+1})-2}{y_1 y_2 \cdots y_k}, & \text{if } 1<k\le n, \end{cases}$$

$$\beta_{k,n} = \begin{cases} \frac{P_-(y_1,\ldots,y_{n+1})}{P_+(y_2,\ldots,y_n)}, & \text{if } k=1, \\ y_2 y_3 \cdots y_k \frac{P_+(y_1,\ldots,y_{n+1})-2}{P_+(y_2,\ldots,y_n)}, & \text{if } 1<k\le n. \end{cases}$$

For $1<k\le n$, let

$$\gamma_{k,n} = \frac{P_+(y_2,\ldots,y_{k-1})}{y_2 \cdots y_k}.$$

Then

$$\mathcal{N}_{k,n} = \begin{cases} [1+y_1;\overline{\alpha_{1,n},\beta_{1,n}}], & \text{if } k=1, \\ [1;\gamma_{k,n},\overline{\beta_{k,n},\alpha_{k,n}}], & \text{if } 1<k\le n. \end{cases} \tag{5.11}$$

One may check that as we expect, under the specialization $y_i \mapsto 1$, we have that for $1 \le k \le n$,

$$\alpha_{k,n} \mapsto n, \beta_{k,n} \mapsto 1,$$

and for $1<k\le n$,

$$\gamma_{k,n} \mapsto k-1.$$

Example 5.4.6. Applying Theorem 5.4.5 to the cases where $n = 2$ and $k = 1, 2$, we obtain

$$\mathcal{N}_{1,2} = \left[1 + y_1; \overline{\frac{1 + y_1 + y_1y_2 - y_1y_2y_3}{y_1^2y_2y_3}, \frac{1 + y_1 + y_1y_2 - y_1y_2y_3}{1 + y_2}}\right],$$

$$\mathcal{N}_{2,2} = \left[1; \frac{1}{y_2}, \overline{\frac{y_2}{1 + y_2}(-1 + y_1 + y_1y_2 + y_1y_2y_3), \frac{-1 + y_1 + y_1y_2 + y_1y_2y_3}{y_1y_2}}\right].$$

In the spirit of the Laurent phenomenon, we might allow generalized continued fractions to obtain expansions that involve only (Laurent) polynomials.

Theorem 5.4.7.

$$\mathcal{N}_{k,n} = \begin{cases} [[1 + y_1, \overline{P_+(y_2, \ldots, y_n), y_1}], [\overline{y_1y_2\cdots y_{n+1}, 1}]] & \text{if } k = 1 \\ [[1; P_+(y_2, \ldots, y_{k-1}), \overline{y_1, P_+(y_2, \ldots, y_n)}], [y_2\cdots y_k, \overline{1, y_1y_2\cdots y_{n+1}}]] & \text{if } 1 < k \le n \end{cases}$$

In Appendix B, we show that the (generalized) infinite continued fraction expansions of $\mathcal{N}_{k,n}$ in Theorems 5.4.5 and 5.4.7 both evaluate to the square root expression of Theorem 5.4.2. As such, Theorems 5.4.2, 5.4.5 and 5.4.7 are equivalent.

Example 5.4.8. We provide in Table 5.3 some of the first few examples of Theorem 5.4.7.

n	k	$[a_0, a_1, \ldots]$	$[b_1, b_2, \ldots]$
1	1	$[1 + y_1; \overline{1, y_1}]$	$[\overline{y_1y_2, 1}]$
2	1	$[1 + y_1; \overline{1 + y_2, y_1}]$	$[\overline{y_1y_2y_3, 1}]$
2	2	$[1; 1, \overline{y_1, 1 + y_2}]$	$[y_2, \overline{1, y_1y_2y_3}]$
3	1	$[1 + y_1; \overline{1 + y_2 + y_2y_3, y_1}]$	$[\overline{y_1y_2y_3y_4, 1}]$
3	2	$[1; 1, \overline{y_1, 1 + y_2 + y_2y_3}]$	$[y_2, \overline{1, y_1y_2y_3y_4}]$
3	3	$[1; 1 + y_2, \overline{y_1, 1 + y_2 + y_2y_3}]$	$[y_2y_3, \overline{1, y_1y_2y_3y_4}]$

Table 5.3 Some examples of continued fraction expansions that involve polynomials only

Remark 5.4.9. When the second way of writing $\mathcal{N}_{k,n}$ as continued fractions was introduced, we wrote that the entries are (Laurent) polynomials rather than polynomials, despite how we never use negative exponents in the statement of Theorem 5.4.7. This is because one could scale all entries in the continued fraction expansions of Theorem 5.4.7 by a certain monomial and obtain essentially equivalent continued fractions which involve Laurent polynomials. For example, compare Table 5.3 with Table 5.4.

n	k	$[a_0, a_1, \ldots]$	$[b_1, b_2, \ldots]$
1	1	$[1 + y_1, \overline{1}]$	$[y_1y_2, \overline{y_1^{-1}, y_2}]$
2	1	$[1 + y_1; \overline{1 + y_2^{-1}, 1}]$	$[y_1y_3, \overline{y_1^{-1}y_2^{-1}, y_3}]$
2	2	$[1; 1, \overline{1, 1 + y_2^{-1}}]$	$[y_2, y_1^{-1}, \overline{y_3, y_1^{-1}y_2^{-1}}]$

Table 5.4 Some examples of continued fraction expansions that involve Laurent polynomials

Rewriting the expressions from Theorem 5.4.7 in the manner of Table 5.4 is motivated the q-continued fractions of Morier-Genoud and Ovsienko (2020), which are q-analogues of classical continued fractions. For example, if we specialize $y_1 = y_2 = q$ in $\mathcal{N}_{1,1}$, we obtain the q-continued fraction

$$\left[\frac{3 + \sqrt{5}}{2}\right]_q = [[1 + q, \overline{1}], [q^2, \overline{q^{-1}, q}]].$$

This connection is detailed in Appendix B.2 of Morier-Genoud and Ovsienko (2020).

5.4.2 First Proof of Theorems 5.4.2, 5.4.5, 5.4.7

In this subsection, we prove the theorem in the form of Theorem 5.4.7. The two key lemmas are Lemma 5.4.11 and Lemma 5.4.14. The proof of Lemma 5.4.11 requires an explicit formula for the $\mathbf{c}, \mathbf{g}$-vectors, which we prove in Proposition 5.4.10. The proof of Lemma 5.4.14 makes use of what are called skein relations, which we briefly introduce. Using these two lemmas, we show that the ratios of F-polynomials satisfy the recurrence of the proposed generalized continued fractions given in Theorem 5.2.3.

We begin by proving the following explicit formulas for the positively-indexed $\mathbf{c}, \mathbf{g}$-vectors.

Proposition 5.4.10. For $k \geq 1$,

$$\mathbf{c}_{k+n+1} = \begin{bmatrix} -\left\lfloor \frac{k+n-1}{n} \right\rfloor \\ -\left\lfloor \frac{k+n-2}{n} \right\rfloor \\ \vdots \\ -\left\lfloor \frac{k-1}{n} \right\rfloor \end{bmatrix}, \quad \mathbf{g}_{k+n+1} = \begin{bmatrix} -\left\lfloor \frac{k+n-1}{n} \right\rfloor \\ \varepsilon_{1,k} \\ \varepsilon_{2,k} \\ \vdots \\ \varepsilon_{n-1,k} \\ \left\lfloor \frac{k}{n} \right\rfloor + 1 \end{bmatrix}, \tag{5.12}$$

where

$$\varepsilon_{i,k} = \begin{cases} 1 & \text{if } k \equiv_n i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. After verifying that the formulas above are correct for $1 \leq k \leq n$, we proceed by induction relying on Equation 2.4 and Equation 2.5. Inspecting the quivers, we find that it suffices to verify that for all $k > n+1$, the following identities are true for the proposed formulas of $\mathbf{c}, \mathbf{g}$-vectors:

$$\mathbf{c}_{k+n+1} + \mathbf{c}_k = \mathbf{c}_{k+1} + \mathbf{c}_{k+n},$$
$$\mathbf{g}_{k+n+1} + \mathbf{g}_k = \mathbf{g}_{k+1} + \mathbf{g}_{k+n},$$

which follows from a direct calculation. □

Lemma 5.4.11. For $1 \leq k \leq n$ and $m \geq 1$,

$$F_{mn+k+1}F_{(m-1)n+k} - F_{mn+k}F_{(m-1)n+k+1} = (y_1 y_2 \cdots y_k)(y_1 y_2 \cdots y_{n+1})^{m-1}.$$

Proof. Let t be the seed that corresponds to the cluster

$$(x_{mn+1}, \ldots, x_{mn+k}, x_{(m-1)n+k}, x_{(m-1)n+k+1}, \ldots, x_{mn}),$$

and let t' be the seed that corresponds to the cluster

$$(x_{mn+1}, \ldots, x_{mn+k}, x_{mn+k+1}, x_{(m-1)n+k+1}, \ldots, x_{mn}).$$

Note that t and t' are both clusters along the source sequence μ_+, and t' is one more mutation away from the initial seed. Now consider the mutation $t \xrightarrow{\mu_{k+1}} t'$. For $1 \leq j \leq 2(n+1)$, let $b^t_{j,k+1}$ denote the $(j, k+1)$-th B-matrix entry at the seed t. By drawing the quiver corresponding to t, we see that there are exactly two outgoing edges from $k+1$ to k and $k+2$ respectively, where the cluster variables are x_{mn+k} and $x_{(m-1)n+k+1}$. In other words, for $1 \le j \le n+1$, $b^t_{j,k+1} = -1$ if $j = k, k+2$, and $b^t_{j,k+1} = 0$ otherwise.

By Proposition 5.4.10, for $1 \le j \le n+1$, we have

$$b^t_{n+1+j,k+1} = \left\lfloor \frac{mn+k-j}{n} \right\rfloor = \begin{cases} m & \text{if } 1 \le j \le k, \\ m-1 & \text{if } k < j \le n+1; \end{cases}$$

there is no negative sign because we have not mutated at $k+1$ yet.

Thus, the exchange rule says that

$$F_{mn+k+1} = \frac{F_{mn+k}F_{(m-1)n+k+1} + (y_1y_2\cdots y_k)(y_1y_2\cdots y_{n+1})^{m-1}}{F_{(m-1)n+k}},$$

which rearranges into the desired identity. □

We provide a simple example to illustrate the lemma.

Example 5.4.12. We always have $F_{n+2} = 1+y_1$ and $F_j = 1$ for $1 \le j \le n+1$. Let $k = 1$ and $m = 1$. Then the theorem says that $F_{n+2}F_1 - F_{n+1}F_2 = y_1$, which is indeed true.

The proofs of the following two lemmas require some background on *skein relations*, or *generalized Ptolemy relations*, which are identities which hold for all cluster algebras of surface type. Given two crossing arcs, there are two *resolutions* of the crossing into two non-crossing arcs. Skein relations are a general identity that involve the cluster variables associated to the crossing arcs and the resolutions. In the setting of principal coefficients, i.e. $y_i = 1$ for all $1 \le i \le n+1$, the skein relation is exactly Figure 5.7, where the picture of two curves represents the product of the cluster variables which correspond to these two curves.

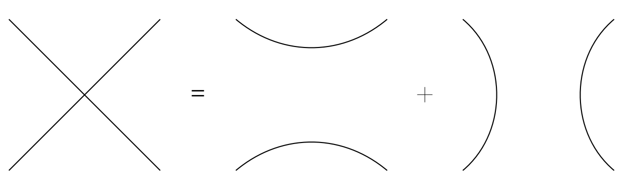


Figure 5.7 Skein relations without coefficients

The skein relations with coefficients can be obtained by adding to one of the two terms in Figure 5.7 an appropriate monomial in the y_i's, which is determined using certain *laminations*. There is a standard collection of laminations of $T_{n,1}$ that we use. When $n = 3$, this is the collection of the gray curves in Figure 5.8. These laminations are constructed from the initial triangulation of $T_{n,1}$ as follows: We perturb the arc that corresponds to x_k in the initial triangulation, so that in the universal cover, the resulting curve crosses the outer (top) boundary slightly to the right of the original crossing, and crosses the inner (bottom) boundary slightly to the left of the original crossing. We do so while making sure that there are no crossings among the resulting laminations. Label by L_k the lamination which comes from the perturbation of x_k. Given a crossing, the two arcs involved together cross some multiset of laminations. One can show that one of the two resolutions crosses the same multiset of laminations, while the other resolution crosses some submultiset of laminations. Moreover, one can show that if the lamination L_k is crossed n_k times by the two crossing arcs, and is crossed m_k times by a resolution of the crossing, then $n_k - m_k$ is even. Let $\varepsilon_k = \frac{n_k - m_k}{2}$. The monomial we add in front of the resolution that crosses fewer laminations is then $\prod_{k=1}^{n+1} y_k^{\varepsilon_k}$.

For a more rigorous and general introduction to skein relations with coefficients, we refer the readers to Musiker and Williams (2013) and Fomin and Thurston (2018).

We name some of the arcs that we need to make use of in the surface $T_{n,1}$. Given the universal cover of $T_{n,1}$ (see Figure 5.8 for the case where $n = 3$), let the peripheral arc from the marked point labeled i to the marked point labeled j be $\alpha_{i,j}$, and let β_j be the bridging arc associated to the F-polynomial F_j. Note that we identify curves on the universal cover which are the same when drawn on $\mathcal{T}_{n,1}$. For example, we say that $\alpha_{4,7} = \alpha_{1,4}$. Under this equivalence, one representative of β_j is the bridging arc whose endpoint on the inner boundary is labeled $1'$, and whose endpoint on the outer boundary is labeled j.

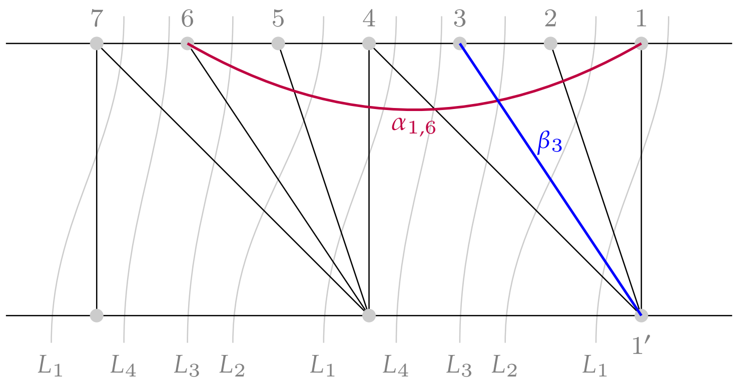


Figure 5.8 The universal cover of $T_{3,1}$, laminations, and examples of $\alpha_{i,j}$ and β_j

Lemma 5.4.13. For $1 < s \leq n+1$, the F-polynomial associated to $\alpha_{1,s}$ is $P_+(y_2, \ldots, y_{s-1})$.

Proof. Let us abuse notation and denote the F-polynomial associated to an arc by the name of the arc. We proceed by induction. When $s = 2$, $\alpha_{1,2} = 1$, which is indeed $P_+(y_2, \ldots, y_1)$. Suppose the claim is true for some $1 < s < n+1$. By applying skein relations to the crossing between $\alpha_{1,s}$ and β_{s-1}, we find that

$$\beta_{s-1}\alpha_{1,s} = \beta_s\alpha_{1,s-1} + y_2y_3\cdots y_{s-1}\beta_1\alpha_{s-1,s}.$$

See Figure 5.9 for an example. Since $\beta_{s-1} = \beta_1 = \alpha_{s-1,s} = 1$, we have

$$\begin{aligned}\alpha_{1,s} &= \alpha_{1,s-1} + y_2y_3\cdots y_{s-1}\\ &= P_+(y_2, \ldots, y_{s-2}) + y_2y_3\cdots y_{s-1}\\ &= P_+(y_2, \ldots, y_{s-1}).\end{aligned}$$

□

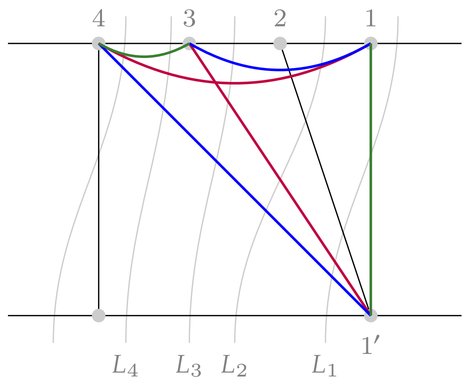


Figure 5.9 When $n = 3, s = 4$, the skein relation associated to the crossing of $\alpha_{1,4}$ and β_3 yields $\alpha_{1,4}\beta_3 = \beta_4\alpha_{1,3} + y_2y_3\beta_1\alpha_{3,4}$

Lemma 5.4.14. Let $1 < s \leq n + 1$. Let $k_0 = 1$, $k_1 = P_+(y_2, \dots, y_{s-1})$,

$$k_m = \begin{cases} P_+(y_2, \dots, y_n)k_{m-1} + y_1y_2\cdots y_{n+1}k_{m-2} & m \text{ is odd,} \\ y_1k_{m-1} + k_{m-2} & m \text{ is even.} \end{cases}$$

Then $k_{2m} = F_{mn+s}(y_1, \dots, y_{n+1})$.

Proof. As before, let us abuse notation and denote the F-polynomial associated to an arc by the name of the arc. It suffices to show that $k_{2m} = \beta_{mn+s}$, and $k_{2m+1} = \alpha_{1,mn+s}$. We prove this by induction.

For our base case, since $F_s = 1$, indeed the F-polynomial associated to the bridging arc β_s is $k_0 = 1$. By Lemma 5.4.13, the F-polynomial associated to the peripheral arc $\alpha_{1,s}$ is indeed $k_1 = P_+(y_2, \dots, y_{s-1})$.

For the inductive step, it suffices to prove two claims: first, for $m > 0$, if $k_{2m} = \beta_{mn+s}$, and $k_{2m-1} = \alpha_{1,(m-1)n+s}$, then $k_{2m+1} = \alpha_{1,mn+s}$; secondly, for $m > 0$, if $k_{2m+1} = \alpha_{1,mn+s}$, $k_{2m} = \beta_{mn+s}$, then $k_{2m+2} = \beta_{(m+1)n+s}$.

To prove the first claim, we consider the crossing of $\alpha_{1,mn+s}$ with β_{n+1}. Some examples are shown in Figure 5.10. The crossing gives rise to the following skein relation:

$$\alpha_{1,mn+s}\beta_{n+1} = \alpha_{1,n+1}\beta_{mn+s} + y_1y_2\cdots y_{n+1}\alpha_{1,(m-1)n+s}\beta_1.$$

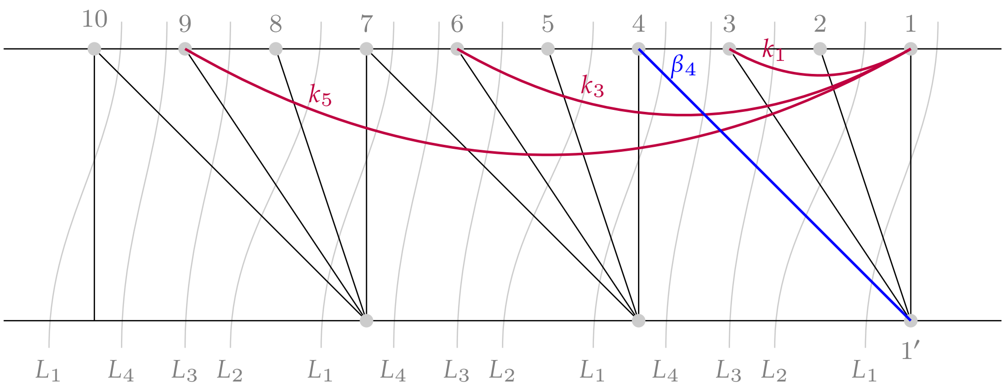


Figure 5.10 When $n = 3$, $s = 3$, the figure shows the arcs associated to k_1, k_3, k_5, as well as the crossings of k_3 and k_5 with β_4, which yields the skein relation that defines them inductively; for instance, $\alpha_{1,6}\beta_4 = \alpha_{1,4}\beta_6 + y_1y_2y_3y_4\alpha_{1,3}\beta_1$

Since $k_{2m-1} = \alpha_{1,(m-1)n+s}$, $\beta_1 = \beta_{n+1} = 1$, $\beta_{mn+s} = k_{2m}$, and $\alpha_{1,n+1} = P_+(y_2, \dots, y_n)$ by Lemma 5.4.13, the above relation simplifies to:

$$\alpha_{1,mn+s} = P_+(y_2, \dots, y_n)k_{2m} + y_1y_2 \cdots y_{n+1}k_{2m-1} = k_{2m+1}.$$

Let us now prove the second claim, for which we consider the crossing of $\beta_{(m+1)n+s}$ with β_1. Some examples are shown in Figure 5.11. Let the boundary arc which winds around the inner boundary once be denoted ω; this is the green segment in Figure 5.11. The crossing gives rise to the following skein relation:

$$\beta_{(m+1)n+s}\beta_1 = y_1\alpha_{1,mn+s}\omega + \beta_{mn+s}\beta_{n+1},$$

which simplifies to

$$\beta_{(m+1)n+s} = y_1k_{2m+1} + k_{2m} = k_{2m+2}.$$

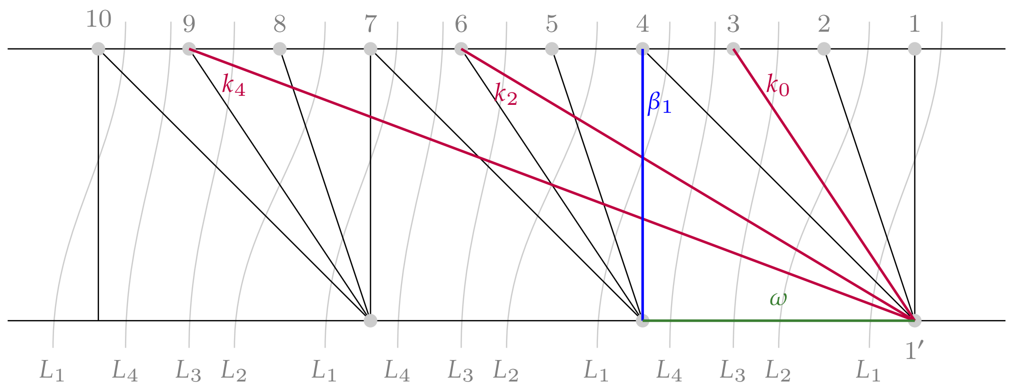


Figure 5.11 When $n = 3$, $s = 3$, the figure shows the arcs associated to k_0, k_2, k_4, as well as the crossings of k_2 and k_4 with β_1, which yields the skein relation that defines them inductively; for instance, $\alpha_{1,6}\beta_4 = \alpha_{1,4}\beta_6 + \alpha_{1,2}\beta_1$

□

We are now ready to prove Theorem 5.4.7.

First Proof. Fix $n \geq 1$ and $1 \leq k \leq n$, and write the continued fraction expansion of $\mathcal{N}_{k,n}$ given by Theorem 5.4.7 as $[[a_0, a_1, a_2, \ldots], [b_1, b_2, \ldots]]$. Then to prove Theorem 5.4.7, it suffices to show that for $1 < k \leq n+1$,

$$\frac{F_{mn+k+1}}{F_{mn+k}} = [[a_0, a_1, \ldots, a_{2m}], [b_1, \ldots, b_{2m}]]. \tag{5.13}$$

Corollary 5.2.4 allows us to proceed by induction. It suffices to show that

1. for $1 < k \leq n+1$, $\frac{F_{k+1}}{F_k} = a_0$. Indeed, $\frac{F_{n+2}}{F_{n+1}} = 1 + y_1$ and $\frac{F_{k+1}}{F_k} = 1$ for $1 < k \leq n$, which is equal to the a_0 in each case;

2. for $m > 1$,

$$\frac{F_{mn+k+1}}{F_{mn+k}} - \frac{F_{(m-1)n+k+1}}{F_{(m-1)n+k}} = \frac{a_{2m} b_1 b_2 \cdots b_{2m-1}}{k_{2m} k_{2m-2}},$$

   where k_m is as given by Theorem 5.2.3.

When $m > 1$, $a_{2m} b_1 b_2 \cdots b_{2m-1} = (y_1 y_2 \cdots y_k)(y_1 y_2 \cdots y_{n+1})^{m-1}$, which, by Lemma 5.4.11, is equal to $F_{mn+k+1} F_{(m-1)n+k} - F_{mn+k} F_{(m-1)n+k+1}$; by Lemma 5.4.14, the denominator can be rewritten as $F_{mn+k} F_{(m-1)n+k}$. This proves the second item, and concludes this proof. □

5.4.3 Second Proof of Theorems 5.4.2, 5.4.5, 5.4.7

In our second proof, we will mostly be working with Theorem 5.4.2, and our strategy is to relate $\mathcal{N}_{k,n}$ for large n to the cases where $n = 1, 2, 3$. We first prove some general results about how F-polynomials and their ratios change as we make a change of initial seed.

Lemma 5.4.15. Let $\mu = \mu_n \mu_{n-1} \cdots \mu_{k_1+1} \mu_2 \mu_3 \cdots \mu_{k_2}$, where $1 \le k_1 \le k_2 \le n$. The sequence μ mutates up to $n-1$ bridging arcs into peripheral arcs, and leaves us bridging arcs labeled by $i_1 < i_2 < \cdots < i_{m+1}$ for some $1 \le m < n$. By construction, $i_1 = 1, i_{m+1} = n+1$, and the subquiver at $i_1, \ldots, i_{m+1}$ of the quiver at the seed $\mu(t)$ is exactly $Q_{m,1}$. There is a unique mutation sequence ν_+ which is a source sequence for the subquiver formed by the bridging arcs. Let $F_\ell(\widehat{y}_1, \ldots, \widehat{y}_{n+1})$ denote the ℓ-th F-polynomial as we apply ν_+ to $\mu(t_o)$. Let $F'_\ell(\widehat{y}_1, \ldots, \widehat{y}_{n+1})$ denote the ℓ-th F-polynomial along ν_+ but written in terms of $\mu(t_o)$ as the initial seed. Let $F''_\ell(\widehat{y}_1, \ldots, \widehat{y}_{m+1})$ denote the ℓ-th F-polynomial of $\mathcal{A}_{m,1}$ along the usual source sequence, starting at the usual initial seed. Let $\widehat{y}' = \mu\widehat{y}$.

Then $F'_\ell(\widehat{y}'_1, \ldots, \widehat{y}'_{n+1}) = F''_\ell(\widehat{y}'_{i_1}, \ldots, \widehat{y}'_{i_{m+1}})$, and

$$\frac{F_{\ell+1}(\widehat{y}_1, \ldots, \widehat{y}_{n+1})}{F_\ell(\widehat{y}_1, \ldots, \widehat{y}_{n+1})} = \frac{F'_{\ell+1}(\widehat{y}'_1, \ldots, \widehat{y}'_{n+1})}{F'_\ell(\widehat{y}'_1, \ldots, \widehat{y}'_{n+1})} = \frac{F''_{\ell+1}(\widehat{y}'_{i_1}, \ldots, \widehat{y}'_{i_{m+1}})}{F''_\ell(\widehat{y}'_{i_1}, \ldots, \widehat{y}'_{i_{m+1}})}.$$

Proof. First observe from the surface model interpretation that along the sequence ν_+, right before we mutate at k, if $b_{jk} \neq 0$ for some j that corresponds to a peripheral arc, we must have $b_{jk} > 0$, whereas k is a source in the subquiver of bridging arcs.

Let $\mathbf{g}_\ell, \mathbf{c}_\ell, \mathbf{g}'_\ell, \mathbf{c}'_\ell$ denote the ℓ-th $\mathbf{g}$, $\mathbf{c}$-vector along ν_+, written in terms of the initial seeds t_o and $\mu(t_o)$ respectively. Due to the observation above, one can show using Equation 2.4 that $\mathbf{c}'_\ell$ is only nonzero at entries i_k and that the sequence $\mathbf{c}'_\ell$ is negative. The fact that $\mathbf{c}'_\ell$ is only nonzero at entries i_k implies that F' and F'' satisfy the same recurrences, and so $F'_\ell(\widehat{y}'_1, \ldots, \widehat{y}'_{n+1}) = F''_\ell(\widehat{y}'_{i_1}, \ldots, \widehat{y}'_{i_{m+1}})$. One can also show, using Equation 2.4, that the $\mathbf{c}_\ell$'s are negative.

Since the $\mathbf{c}_\ell$'s are negative, and also using the first observation, the second recurrence of $\mathbf{g}$-vectors in Equation 2.5 never involves $\mathbf{g}_j$ where $j \neq i_k$. This implies that $\mathbf{g}_\ell = \mathbf{g}'_\ell$ for all $\ell > 0$. Since $\mathbf{g}_{i_k} = \mathbf{e}_{i_k}$, we have that the $\mathbf{g}_\ell$ vectors are only nonzero at entries i_k, where $1 \le k \le m+1$.

In general, for a cluster variable x_α with $\mathbf{g}$-vector $\mathbf{g}_\alpha$ and F-polynomial

F_α, we have

$$x_\alpha = \mathbf{x}^{\mathbf{g}_\alpha} F_\alpha(\widehat{y}).$$

It follows that

$$\frac{F_\ell(\widehat{y})}{F'_\ell(\widehat{y}')} = \frac{\mathbf{x}'^{\mathbf{g}'_\ell}}{\mathbf{x}^{\mathbf{g}_\ell}}.$$

But our previous discussion implies that the right hand side is 1. Therefore,

$$\frac{F_{\ell+1}(\widehat{y}_1, \ldots, \widehat{y}_{n+1})}{F_\ell(\widehat{y}_1, \ldots, \widehat{y}_{n+1})} = \frac{F'_{\ell+1}(\widehat{y}'_1, \ldots, \widehat{y}'_{n+1})}{F'_\ell(\widehat{y}'_1, \ldots, \widehat{y}'_{n+1})}.$$

□

The following lemma tells us what $\widehat{y}'$ is.

Lemma 5.4.16. Let $(\widehat{y}_1, \ldots, \widehat{y}_{n+1}) = (\widehat{y}_{1;t_o}, \ldots, \widehat{y}_{n+1;t_o})$, where t_o is the customary initial seed of $\widetilde{A}_{n,1}$. For $1 \le k \le n-1$, let $\mu = \mu_n \mu_{n-1} \cdots \mu_{k+1}$. Then

$$\begin{aligned}
\mu(\widehat{y}_j) &= \widehat{y}_j \text{ for } j < k, \\
\mu(\widehat{y}_k) &= \widehat{y}_k P_+(\widehat{y}_{k+1}, \ldots, \widehat{y}_n), \\
\mu(\widehat{y}_{n+1}) &= \frac{\widehat{y}_{k+1}\widehat{y}_{k+2} \cdots \widehat{y}_{n+1}}{P_+(\widehat{y}_{k+1}, \ldots, \widehat{y}_n)}.
\end{aligned}$$

Similarly, for $2 \le k \le n$, let $\nu = \mu_2 \mu_3 \cdots \mu_k$. Then

$$\begin{aligned}
\nu(\widehat{y}_1) &= \widehat{y}_1 P_+(\widehat{y}_2, \ldots, \widehat{y}_k), \\
\nu(\widehat{y}_{k+1}) &= \frac{\widehat{y}_2\widehat{y}_3 \cdots \widehat{y}_{k+1}}{P_+(\widehat{y}_2, \ldots, \widehat{y}_k)}. \\
\nu(\widehat{y}_j) &= \widehat{y}_j \text{ for } j > k+1.
\end{aligned}$$

Proof. We prove the first part of the lemma since the second part is analogous. We proceed by induction. Let $\mu(n-k) = \mu_n \mu_{n-1} \cdots \mu_{k+1}$, and let $t_{n-k} = \mu(n-k)t_o$. When $k = n-1$, since $b^{t_o}_{n,n+1} = 1$, $b^{t_o}_{n,n-1} = -1$, and $b^{t_o}_{n,j} = 0$ for other $1 \le j \le n+1$, indeed Equation 2.6 suggests that $\widehat{y}_{j;t_1} = \widehat{y}_{j;t_o}$ for $j < k$, and

$$\begin{aligned}
\widehat{y}_{n-1;t_1} &= \widehat{y}_{n-1;t_o}(1 + \widehat{y}_{n;t_o}) = \widehat{y}_{n-1}(1 + \widehat{y}_n), \\
\widehat{y}_{n+1;t_1} &= \widehat{y}_{n+1;t_o}\widehat{y}_{n;t_o}(1 + \widehat{y}_{n;t_o})^{-1} = \frac{\widehat{y}_n \widehat{y}_{n+1}}{P_+(\widehat{y}_n)},
\end{aligned}$$

as the lemma suggests.

Assuming the lemma is true for some $1 < k \le n-1$, we want to show that it is also true for $k-1$. Consider the mutation μ_k from t_{n-k} to t_{n-k+1}. We observe that $b^{t_{n-k}}_{k,n+1} = 1, b^{t_{n-k}}_{k,k-1} = -1$, and $b^{t_{n-k}}_{k,j} = 0$ for other $1 \le j \le n+1$. So by Equation 2.6,

$$\begin{aligned}
\widehat{y}_{k-1;t_{n-k+1}} &= \widehat{y}_{k-1;t_{n-k}}(1+\widehat{y}_{k;t_{n-k}}) \\
&= \widehat{y}_{k-1}(1+\widehat{y}_k P_+(\widehat{y}_{k+1},\dots,\widehat{y}_n)) \\
&= \widehat{y}_{k-1}P_+(\widehat{y}_k,\dots,\widehat{y}_n), \\
\widehat{y}_{n+1;t_{n-k+1}} &= \widehat{y}_{n+1;t_{n-k}}\widehat{y}_{k;t_{n-k}}(1+\widehat{y}_{k;t_{n-k}})^{-1} \\
&= \frac{\widehat{y}_{k+1}\cdots\widehat{y}_{n+1}}{P_+(\widehat{y}_{k+1},\dots,\widehat{y}_n)}\,\frac{\widehat{y}_k P_+(\widehat{y}_{k+1},\dots,\widehat{y}_n)}{1+\widehat{y}_k P_+(\widehat{y}_{k+1},\dots,\widehat{y}_n)} \\
&= \frac{\widehat{y}_k\cdots\widehat{y}_{n+1}}{P_+(\widehat{y}_k,\dots,\widehat{y}_n)}.
\end{aligned}$$

By our induction hypothesis, $\mu(n-k)(\widehat{y}_j) = \widehat{y}_j$ for $j < k$. Thus, for $1 \le j < k-1$, since $b^{t_{n-k}}_{k,j} = 0$, indeed we have

$$\mu(n-k+1)(\widehat{y}_j) = \mu_k(\mu(n-k)(\widehat{y}_j)) = \mu_k(\widehat{y}_j) = \widehat{y}_j.$$

□

This concludes our discussion about changes of initial seeds in $\mathcal{A}_{n,1}$. Now we are ready to use some of these results towards proving Theorem 5.4.2. As we saw in Section 5.3, Theorem 5.4.2 is already proven for $\mathcal{N}_{1,1}$. An appropriate change of seed helps us obtain the following theorem from the formula for $\mathcal{N}_{1,1}$.

Theorem 5.4.17. Let

$$\mathcal{T}_n(\widehat{y}) = \prod_{k=1}^{n}\mathcal{N}_{k,n}(\widehat{y}) = \lim_{m\to\infty}\frac{F_{m+n}(\widehat{y}_1,\dots,\widehat{y}_{n+1})}{F_m(\widehat{y}_1,\dots,\widehat{y}_{n+1})}.$$

Then

$$\mathcal{T}_n = \frac{1}{2}\left(P_+(\widehat{y}_1,\dots,\widehat{y}_{n+1}) + \sqrt{\Delta_n}\right),$$

where $\Delta_n = P_-(\widehat{y}_1,\dots,\widehat{y}_{n+1})^2 + 4\widehat{y}_1^2\widehat{y}_2\cdots\widehat{y}_{n+1}P_+(\widehat{y}_2,\cdots,\widehat{y}_n)$.

Proof. We first point out a subtlety in our definition of $\mathcal{T}_n$. Because

$$\mathcal{T}_n = \prod_{i=1}^{n} \mathcal{N}_{i,n},$$

by rearranging the order of this product, we can show that

$$\mathcal{T}_n(\widehat{y}) = \lim_{m\to\infty} \frac{F_{(k+1)n+i}(\widehat{y}_1, \dots, \widehat{y}_{n+1})}{F_{kn+i}(\widehat{y}_1, \dots, \widehat{y}_{n+1})}$$

for all $0 \leq i < n$.

Thus, it suffices to consider the limit

$$\lim_{m\to\infty} \frac{F_{(k+1)n+1}(\widehat{y}_1, \dots, \widehat{y}_{n+1})}{F_{(k+1)n}(\widehat{y}_1, \dots, \widehat{y}_{n+1})}.$$

Let $\mu = \mu_n \mu_{n-1} \dots \mu_2$. By Lemma 5.4.16, we know that

$$\widehat{y}'_1 = \mu(\widehat{y}_1) = \widehat{y}_1 P_+(\widehat{y}_2, \dots, \widehat{y}_n),\ \ \widehat{y}'_{n+1} = \mu(\widehat{y}_{n+1}) = \frac{\widehat{y}_2\widehat{y}_3 \cdots \widehat{y}_{n+1}}{P_+(\widehat{y}_2, \dots, \widehat{y}_n)}.$$

Taking the limit of the identity in Lemma 5.4.15, we get that

$$\begin{aligned}
\mathcal{T}_n &= \lim_{m\to\infty} \frac{F_{m+n}(y_1, \dots, y_{n+1})}{F_m(y_1, \dots, y_{n+1})} \\
&= \mathcal{N}(\widehat{y}'_1, \widehat{y}'_{n+1}) \\
&= \frac{1}{2}\left(1 + \widehat{y}'_1 + \widehat{y}'_1\widehat{y}'_{n+1} + \sqrt{(1 + \widehat{y}'_1 - \widehat{y}'_1\widehat{y}'_{n+1})^2 + 4\widehat{y}'^2_1\widehat{y}'_{n+1}}\right).
\end{aligned}$$

which simplifies to the desired expression for $\mathcal{T}_n$. □

Knowing $\mathcal{T}_2$, we are ready to prove Theorem 5.4.2 for $\mathcal{N}_{1,2}$.

Lemma 5.4.18.

$$\mathcal{N}_{1,2} = \frac{1 + y_1 + y_1y_2 + y_1y_2y_3 + 2y_2 + \sqrt{\Delta_2}}{2(1 + y_2)}$$

Proof. Let $w = y_2 + 1$ be the F-polynomial of the cluster variable obtained by mutating at x_{2k} in the cluster $\{x_{2k-1}, x_{2k}, x_{2k+1}\}$. In the surface $T_{2,1}$, w corresponds to the peripheral arc whose endpoints are both the bottom marked point on the outer boundary. Then the exchange relation tells us that

$$w = \frac{F_{2k+1} + y_2F_{2k-1}}{F_{2k}},$$

which reorganizes into

$$F_{2k+1} - F_{2k} = y_2(F_{2k} - F_{2k-1}).$$

Equivalently, we have

$$\frac{F_{2k+1}}{F_{2k}} - 1 = y_2(1 - \frac{F_{2k+1}}{F_{2k}} / \frac{F_{2k+1}}{F_{2k-1}}).$$

If we take $k \to \infty$, we obtain

$$\mathcal{N}_{2,2} - 1 = y_2 \left(1 - \frac{\mathcal{N}_{2,2}}{\mathcal{T}_2}\right).$$

By Theorem 5.4.17,

$$\mathcal{T}_2 = \frac{1}{2} \left(P_+(y_1, y_2, y_3) + \sqrt{\Delta_2}\right).$$

It follows that

$$\mathcal{N}_{2,2} = \frac{1 + y_2}{1 + y_2/\mathcal{T}_2} = \frac{(1 + y_2)(P_+(y_1, y_2, y_3) + \sqrt{\Delta_2})}{P_+(y_1, y_2, y_3) + 2y_2 + \sqrt{\Delta_2}}.$$

If we simplify this expression by multiplying both the denominator and numerator by $P_+(y_1, y_2, y_3) + 2y_2 - \sqrt{\Delta_2}$, we obtain the desired formula for $\mathcal{N}_{2,2}$. □

Knowing $\mathcal{N}_{1,2}$ allows us to verify Theorem 5.4.2 for the cases where $k = 1, n + 1$.

Lemma 5.4.19. Theorem 5.4.2 holds when $k = 1, n + 1$.

Proof. We prove the case where $k = 1$. The case where $k = n+1$ is analogous. Let $\mu = \mu_n \mu_{n-1} \cdots \mu_3$ and let $\widehat{y}' = \mu(\widehat{y})$. By Lemma 5.4.16,

$$\widehat{y}'_1 = \widehat{y}_1, \widehat{y}'_2 = \widehat{y}_2 P_+(\widehat{y}_3, \ldots, \widehat{y}_n), \widehat{y}'_{n+1} = \frac{\widehat{y}_3 \cdots \widehat{y}_{n+1}}{P_+(\widehat{y}_2, \ldots, \widehat{y}_n)}.$$

Taking the limit of the identity in Lemma 5.4.15, we obtain

$$\mathcal{N}_{1,n} = \mathcal{N}_{1,2}(\widehat{y}'_1, \widehat{y}'_2, \widehat{y}'_{n+1}),$$

which simplifies to the desired formula for $\mathcal{N}_{1,n}$. □

We are now ready to show that Theorem 5.4.2 holds for $k = 2$ and $n = 3$.

Lemma 5.4.20.

$$\mathcal{N}_{2,3}(y_1, y_2, y_3, y_4) = \frac{P_+(y_1, \ldots, y_4) + 2y_3(1 + y_1 + y_1 y_4) + \sqrt{\Delta_3}}{2(1 + y_1 + y_3 + y_1 y_3 + y_1 y_3 y_4)},$$

where $\Delta_3 = P_-(y_1, \ldots, y_4)^2 + 4y_1^2 y_2 y_3 y_4(1 + y_2 + y_2 y_3)$.

Proof. Theorem 5.4.17 gives us a formula for $\mathcal{T}_3 = \mathcal{N}_{1,3}\mathcal{N}_{2,3}\mathcal{N}_{3,3}$. Lemma 5.4.19 gives us a formula for $\mathcal{N}_{1,3}$ and $\mathcal{N}_{3,3}$. After some algebra, one verifies that $\mathcal{N}_{2,3} = \frac{\mathcal{T}_3}{\mathcal{N}_{1,3}\mathcal{N}_{3,3}}$ is given by the formula above. $\square$

We are now ready to prove Theorem 5.4.2 for general $n \geq 1$ and $1 \leq k \leq n$.

Second Proof of Theorem 5.4.2. In light of Lemma 5.4.19, it remains to prove Theorem 5.4.2 when $1 < k < n$. To compute $\mathcal{N}_{k,n}$, we apply Lemma 5.4.16 and 5.4.15 to the sequence $\mu_n \mu_{n-1} \cdots \mu_{k+2} \mu_2 \mu_3 \cdots \mu_{k-1}$ to obtain

$$\mathcal{N}_{k,n} = \mathcal{N}_{2,3}(\widehat{y}'_1, \widehat{y}'_k, \widehat{y}'_{k+1}, \widehat{y}'_{n+1}),$$

which one may verify indeed simplifies to the formula given in Theorem 5.4.2. $\square$

5.4.4 Gupta's Formula and Continued Fractions

Yet another way to understand the power series $\mathcal{N}_{k,n}$ is to use Gupta's formula to write it as an infinite product of rational functions in $\widehat{y}$, or as a sum over infinite integer sequences of monomials. To do so, we will first compute the F-polynomials of $\widetilde{A_{n,1}}$ along μ_+. We acknowledge that a lot of the calculations below were done implicitly in Gupta (2018). To match with the convention of Gupta's formula, for $k \geq 1$, Throughout this subsection, we work with the mutation sequence μ_+, and define $\mathbf{g}_k$, $\mathbf{c}_k$ in the same manner as Theorem 3.1.1. This shifts the indexing introduced in Section 5.1 by $n + 1$. The following proposition restates Proposition 5.4.10 in this new indexing scheme.

Proposition 5.4.21. Recall that $\mathbf{g}_k, \mathbf{c}_k$ denote the $\mathbf{g}$-vector and $\mathbf{c}$-vector of x_k. Then for $k \geq 1$,

$$\mathbf{c}_k = \begin{bmatrix} -\left\lfloor \frac{k+n-1}{n} \right\rfloor \\ -\left\lfloor \frac{k+n-2}{n} \right\rfloor \\ \vdots \\ -\left\lfloor \frac{k-1}{n} \right\rfloor \end{bmatrix}, \quad \mathbf{g}_k = \begin{bmatrix} -\left\lfloor \frac{k+n-1}{n} \right\rfloor \\ \varepsilon_{1,k} \\ \varepsilon_{2,k} \\ \vdots \\ \varepsilon_{n-1,k} \\ \left\lfloor \frac{k}{n} \right\rfloor + 1 \end{bmatrix}, \tag{5.14}$$

where

$$\varepsilon_{i,k} = \begin{cases} 1 & \text{if } k \equiv_n i, \\ 0 & \text{otherwise.} \end{cases}$$

□

Recall that B_Q denotes the initial (square) exchange matrix of the cluster algebra defined by the quiver Q, which equals the signed adjacency matrix of Q. With Proposition 5.4.21, we can calculate $\mathbf{c}_j \cdot B_{Q_{n,1}} |\mathbf{c}_k|$, which is used in Gupta's formula.

Proposition 5.4.22. For $j, k \geq 1$,

$$\mathbf{c}_j \cdot B_{Q_{n,1}} |\mathbf{c}_k| = \left\lfloor \frac{j+n-1-k}{n} \right\rfloor + \left\lfloor \frac{j-k}{n} \right\rfloor.$$

Proof. Write $S_{i,j}$ for the $(n+1)$-by-$(n+1)$ matrix whose only nonzero entries are the (i,j)-th entry, where it is equal to 1, and the (j,i)-th entry, where it is equal to -1. Then by the definitions of $Q_{n,1}$,

$$B_{Q_{n,1}} = S_{1,n+1} + S_{1,2} + S_{2,3} + \cdots + S_{n,n+1}.$$

Also let κ, ν be the unique integers such that $k = \kappa n + \nu$ and $0 < \nu \leq n$. Notice that $\kappa = \left\lfloor \frac{k-1}{n} \right\rfloor$. Using Proposition 5.4.10, we calculate that

$$\begin{aligned} \mathbf{c}_j \cdot S_{1,n+1} |\mathbf{c}_k| &= -\left\lfloor \frac{j+n-1}{n} \right\rfloor \left\lfloor \frac{k-1}{n} \right\rfloor + \left\lfloor \frac{j-1}{n} \right\rfloor \left\lfloor \frac{k+n-1}{n} \right\rfloor \\ &= \left\lfloor \frac{j-1}{n} \right\rfloor - \kappa. \end{aligned}$$

Also, for $0 \leq m < n$,

$$\mathbf{c}_j \cdot S_{n-m,n-m+1} |\mathbf{c}_k| = -\left\lfloor \frac{j+m}{n} \right\rfloor \left\lfloor \frac{k+m-1}{n} \right\rfloor + \left\lfloor \frac{j+m-1}{n} \right\rfloor \left\lfloor \frac{k+m}{n} \right\rfloor.$$

We now compute the sum $\sum_{m=0}^{n-1} \mathbf{c}_j \cdot S_{n-m,n-m+1}\,|\mathbf{c}_k|$. Notice that for $0 \leq m < n$,

$$\left\lfloor \frac{k+m}{n} \right\rfloor = \begin{cases} \kappa+1 & \text{if } m \geq n-\nu, \\ \kappa & \text{if } m < n-\nu. \end{cases}$$

So depending on m, each $\mathbf{c}_j \cdot S_{n-m,n-m+1}\,|\mathbf{c}_k|$ is of one of the following forms:

$$-\left\lfloor \frac{j+m}{n} \right\rfloor (\kappa+1) + \left\lfloor \frac{j+m-1}{n} \right\rfloor (\kappa+1),$$

$$-\left\lfloor \frac{j+m}{n} \right\rfloor \kappa + \left\lfloor \frac{j+m-1}{n} \right\rfloor (\kappa+1),$$

and

$$-\left\lfloor \frac{j+m}{n} \right\rfloor \kappa + \left\lfloor \frac{j+m-1}{n} \right\rfloor \kappa.$$

We now divide into two cases. In the first case, $\nu \neq 1$. Then

$$\begin{aligned}
&\sum_{m=0}^{n-1} \mathbf{c}_j \cdot S_{n-m,n-m+1}\,|\mathbf{c}_k| \\
&= \kappa\left(-\left\lfloor \frac{j+n-1}{n} \right\rfloor + \left\lfloor \frac{j+n-2}{n} \right\rfloor - \left\lfloor \frac{j+n-2}{n} \right\rfloor + \left\lfloor \frac{j+n-3}{n} \right\rfloor - \cdots - \left\lfloor \frac{j}{n} \right\rfloor + \left\lfloor \frac{j-1}{n} \right\rfloor\right) \\
&\quad + \left(-\left\lfloor \frac{j+n-1}{n} \right\rfloor + \left\lfloor \frac{j+n-2}{n} \right\rfloor - \cdots - \left\lfloor \frac{j+n-\nu+1}{n} \right\rfloor + \left\lfloor \frac{j+n-\nu}{n} \right\rfloor + \left\lfloor \frac{j+n-\nu-1}{n} \right\rfloor\right) \\
&= \kappa\left(-\left\lfloor \frac{j+n-1}{n} \right\rfloor + \left\lfloor \frac{j-1}{n} \right\rfloor\right) + \left(-\left\lfloor \frac{j+n-1}{n} \right\rfloor + \left\lfloor \frac{j+n-\nu}{n} \right\rfloor + \left\lfloor \frac{j+n-\nu-1}{n} \right\rfloor\right) \\
&= -\kappa - \left\lfloor \frac{j+n-1}{n} \right\rfloor + \left\lfloor \frac{j+n-\nu}{n} \right\rfloor + \left\lfloor \frac{j+n-\nu-1}{n} \right\rfloor.
\end{aligned}$$

In the second case, $\nu = 1$. In this case, the sum $\sum_{m=0}^{n-1} \mathbf{c}_j \cdot S_{n-m,n-m+1}\,|\mathbf{c}_k|$ is slightly different because $\left\lfloor \frac{k+m}{n} \right\rfloor = \kappa+1$ only when $m = n-1$. By a similar method, we get that

$$\sum_{m=0}^{n-1} \mathbf{c}_j \cdot S_{n-m,n-m+1}\,|\mathbf{c}_k| = -\kappa + \left\lfloor \frac{j+n-2}{n} \right\rfloor.$$

But this turns out to agree with our calculation in the first case if we substitute $\nu = 1$. So in both cases,

$$\begin{aligned}
\mathbf{c}_j \cdot B_{Q_{n,1}} |\mathbf{c}_k| &= \mathbf{c}_j \cdot S_{1,n+1} |\mathbf{c}_k| + \sum_{m=0}^{n-1} \mathbf{c}_j \cdot S_{n-m,n-m+1} |\mathbf{c}_k| \\
&= \left\lfloor \frac{j-1}{n} \right\rfloor - \kappa - \kappa - \left\lfloor \frac{j+n-1}{n} \right\rfloor + \left\lfloor \frac{j+n-\nu}{n} \right\rfloor + \left\lfloor \frac{j+n-\nu-1}{n} \right\rfloor \\
&= \left\lfloor \frac{j+n-\nu-1}{n} \right\rfloor + \left\lfloor \frac{j-\nu}{n} \right\rfloor - 2\kappa.
\end{aligned}$$

Substituting $k = \kappa n + \nu$, we may verify the following identity,

$$\left\lfloor \frac{j+n-1-\nu}{n} \right\rfloor + \left\lfloor \frac{j-\nu}{n} \right\rfloor - 2\kappa = \left\lfloor \frac{j+n-1-k}{n} \right\rfloor + \left\lfloor \frac{j-k}{n} \right\rfloor.$$

This concludes the proof. □

Example 5.4.23 (Kronecker)**.** When $n = 1$,

$$\mathbf{c}_j \cdot B_{Q_{n,1}} |\mathbf{c}_k| = j - k + j - k = 2(j-k).$$

Example 5.4.24 ($\widetilde{A}_{2,1}$)**.** When $n = 2$,

$$\mathbf{c}_j \cdot B_{Q_{n,1}} |\mathbf{c}_k| = \left\lfloor \frac{j+1-k}{2} \right\rfloor + \left\lfloor \frac{j-k}{2} \right\rfloor.$$

When $k = j$, this evaluates to 0. Inspecting the sum also suggests that $\mathbf{c}_j \cdot B_{Q_{n,1}} |\mathbf{c}_k| - \mathbf{c}_j \cdot B_{Q_{n,1}} |\mathbf{c}_{k+1}| = 1$. Therefore, by induction,

$$\left\lfloor \frac{j+1-k}{2} \right\rfloor + \left\lfloor \frac{j-k}{2} \right\rfloor = j - k.$$

Proposition 5.4.25. Let $j, \ell \geq 1$. Then

$$\mathbf{c}_j \cdot \mathbf{g}_\ell = \left\lfloor \frac{\ell-1}{n} \right\rfloor - \sum_{i=1}^{n} \varepsilon_{i,\ell} \left\lfloor \frac{j-1-i}{n} \right\rfloor = -\left\lfloor \frac{j-\ell-1}{n} \right\rfloor = \left\lceil \frac{\ell+1-j}{n} \right\rceil.$$

Proof. By Proposition 5.4.10, for $j, \ell \geq 1$,

$$\mathbf{c}_j \cdot \mathbf{g}_\ell = \begin{bmatrix} -\left\lfloor \frac{j+n-1}{n} \right\rfloor \\ -\left\lfloor \frac{j+n-2}{n} \right\rfloor \\ -\left\lfloor \frac{j+n-3}{n} \right\rfloor \\ \vdots \\ -\left\lfloor \frac{j}{n} \right\rfloor \\ -\left\lfloor \frac{j-1}{n} \right\rfloor \end{bmatrix} \cdot \begin{bmatrix} -\left\lfloor \frac{\ell+n-1}{n} \right\rfloor \\ \varepsilon_{1,\ell} \\ \varepsilon_{2,\ell} \\ \vdots \\ \varepsilon_{n-1,\ell} \\ \left\lfloor \frac{\ell}{n} \right\rfloor + 1 \end{bmatrix}$$

$$= \left\lfloor \frac{j+n-1}{n} \right\rfloor \left\lfloor \frac{\ell+n-1}{n} \right\rfloor - \left\lfloor \frac{j-1}{n} \right\rfloor \left\lfloor \frac{\ell}{n} \right\rfloor - \left\lfloor \frac{j-1}{n} \right\rfloor - \sum_{i=1}^{n-1} \varepsilon_{i,\ell} \left\lfloor \frac{j-1-i}{n} \right\rfloor$$

$$= \left\lfloor \frac{\ell-1}{n} \right\rfloor - \left\lfloor \frac{j-1}{n} \right\rfloor \left(\left\lfloor \frac{\ell}{n} \right\rfloor - \left\lfloor \frac{\ell-1}{n} \right\rfloor \right) - \sum_{i=1}^{n-1} \varepsilon_{i,\ell} \left\lfloor \frac{j-1-i}{n} \right\rfloor$$

But notice that $\varepsilon_{n,\ell} = \left\lfloor \frac{\ell}{n} \right\rfloor - \left\lfloor \frac{\ell-1}{n} \right\rfloor$, so

$$\begin{aligned} \mathbf{c}_j \cdot \mathbf{g}_\ell &= \left\lfloor \frac{\ell-1}{n} \right\rfloor - \sum_{i=1}^{n} \varepsilon_{i,\ell} \left\lfloor \frac{j-1-i}{n} \right\rfloor \\ &= -\left\lfloor \frac{j-\ell-1}{n} \right\rfloor \\ &= \left\lceil \frac{\ell+1-j}{n} \right\rceil. \end{aligned}$$

As in the proof of Proposition 5.4.22, the second equality above can be verified by a straightforward substitution where we let $\ell = \kappa n + \nu$, where $0 < \nu \leq n$. □

We can now apply Gupta's formula (Theorem 3.1.1 and Lemma 3.3.1) to obtain formulas of F-polynomials and ratios of F-polynomials in $\widetilde{A_{n,1}}$.

Theorem 5.4.26. Let $\mu = \mu_+(\ell)$ and let L_j be defined as in Theorem 3.1.1.

Then in $\widetilde{A}_{n,1}$, for $\ell \geq 1$,

$$F_\ell(y_1,\ldots,y_{n+1}) = \prod_{j=1}^{\ell} L_j^{\left\lceil \frac{\ell+1-j}{n} \right\rceil}$$

$$= \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}^{\ell}} \prod_{j=1}^{\ell} \binom{\left\lceil \frac{\ell+1-j}{n} \right\rceil + \sum_{k=j+1}^{\ell}\left(\left\lfloor \frac{j+n-1-k}{n}\right\rfloor + \left\lfloor \frac{j-k}{n} \right\rfloor\right) m_k}{m_j}$$

$$y_1^{\sum_{k=1}^{\ell}\left\lfloor \frac{k+n-1}{n}\right\rfloor m_k} y_2^{\sum_{k=1}^{\ell}\left\lfloor \frac{k+n-2}{n}\right\rfloor m_k} \cdots y_{n+1}^{\sum_{k=1}^{\ell}\left\lfloor \frac{k-1}{n}\right\rfloor m_k}.$$

This implies that

$$\frac{F_{\ell+1}(y_1,\ldots,y_{n+1})}{F_\ell(y_1,\ldots,y_{n+1})} = \prod_{\substack{1\leq j\leq \ell+1 \\ j\equiv_n \ell+1}} L_j$$

$$= \prod_{j=1}^{\ell+1} L_j^{\varepsilon_{j,\ell+1}}$$

$$= \sum_{(m_1,\ldots,m_{\ell+1})\in\mathbb{Z}_{\geq 0}^{\ell+1}} \prod_{j=1}^{\ell+1} \binom{\varepsilon_{j,\ell+1} + \sum_{k=j+1}^{\ell+1}\left(\left\lfloor \frac{j+n-1-k}{n}\right\rfloor + \left\lfloor \frac{j-k}{n} \right\rfloor\right) m_k}{m_j}$$

$$y_1^{\sum_{k=1}^{\ell+1}\left\lfloor \frac{k+n-1}{n}\right\rfloor m_k} y_2^{\sum_{k=1}^{\ell+1}\left\lfloor \frac{k+n-2}{n}\right\rfloor m_k} \cdots y_{n+1}^{\sum_{k=1}^{\ell+1}\left\lfloor \frac{k-1}{n}\right\rfloor m_k}$$

Example 5.4.27. When $n = 1$, the above calculation specializes to

$$\frac{F_{\ell+1}(y_1,\ldots,y_{n+1})}{F_\ell(y_1,\ldots,y_{n+1})} = \prod_{j=1}^{\ell+1} L_j$$

$$= \sum_{(m_1,\ldots,m_{\ell+1})\in\mathbb{Z}_{\geq 0}^{\ell+1}} \prod_{j=1}^{\ell+1} \binom{1 + \sum_{k=j+1}^{\ell+1} 2(j-k)m_k}{m_j} y_1^{\sum_{k=1}^{\ell+1} k m_k} y_2^{\sum_{k=1}^{\ell+1}(k-1)m_k}$$

which is Equation 5.9 from Section 5.3.

Let us also consider $\widetilde{A}_{2,1}$. By the calculation in Example 5.4.24 and

Theorem 5.4.26, we have that for $\ell \geq 1$,

$$\begin{aligned}\frac{F_{2\ell+1}}{F_{2\ell}} &= L_1L_3\dots L_{2\ell+1}\\ &= \sum_{(m_1,m_2,\dots,m_{2\ell+1})\in\mathbb{Z}_{\geq 0}^{2\ell+1}} \prod_{j=1}^{2\ell+1}\binom{\varepsilon_{j,1}+\sum_{k=j+1}^{2\ell+1}(j-k)m_k}{m_j}\\ &\quad y_1^{\sum_{k=1}^{2\ell+1}\lfloor\frac{k+1}{2}\rfloor m_k} y_2^{\sum_{k=1}^{2\ell+1}\lfloor\frac{k}{2}\rfloor m_k} y_3^{\sum_{k=1}^{2\ell+1}\lfloor\frac{k-1}{2}\rfloor m_k}.\end{aligned}$$

For instance,

$$\begin{aligned}\frac{F_5}{F_4} &= L_1L_3L_5\\ &= \textstyle\sum_{(m_1,m_2,\dots,m_5)\in\mathbb{Z}_{\geq 0}^5}\binom{1-m_2-2m_3-3m_4-4m_5}{m_1}\binom{-m_3-2m_4-3m_5}{m_2}\binom{1-m_4-2m_5}{m_3}\binom{-m_5}{m_4}\binom{1}{m_5}y_1^M y_2^N\end{aligned}$$

where

$$M = m_1+m_2+2m_3+2m_4+3m_5,\ \ N = m_2+m_3+2m_4+2m_5.$$

The combination of Theorem 5.4.7 and Gupta's formula brings an interesting interpretation of the L_i's in Gupta's formula to the table, namely that they are ratios of every other continuant of the appropriate generalized infinite continued fraction. We illustrate this with the simplest example, $\mathcal{N}_{1,1}$. Let L_i be the factors from Theorem 3.1.1, where $\mu = \mu_+ = \mu_1\mu_2\mu_1\mu_2\cdots$. As demonstrated in the first proof of Theorem 5.4.7, the sequence of ratios $\frac{F_{\ell+1}}{F_\ell}$ can be written as a sequence of continuants, of which the following are the first few terms:

$$\begin{aligned}\frac{F_3}{F_2} &= [[1+y_1],[]],\\ \frac{F_4}{F_3} &= [[1+y_1,1,y_1],[y_1y_2,1,y_1y_2,1]],\\ \frac{F_5}{F_4} &= [[1+y_1,1,y_1,1,y_1],[y_1y_2,1,y_1y_2,1,y_1y_2,1]].\end{aligned}$$

But we also know that

$$\frac{F_{\ell+1}}{F_\ell} = L_1L_2\cdots L_{\ell+1}.$$

Therefore,

$$\begin{aligned}
L_1 &= \frac{F_3/F_2}{F_2/F_1} = [[1+y_1],[]],\\
L_2 &= \frac{F_4/F_3}{F_3/F_2} = \frac{[[1+y_1,1,y_1],[y_1y_2,1,y_1y_2,1]]}{[[1+y_1],[]]},\\
L_3 &= \frac{F_5/F_4}{F_4/F_3} = \frac{[[1+y_1,1,y_1,1,y_1],[y_1y_2,1,y_1y_2,1,y_1y_2,1]]}{[[1+y_1,1,y_1],[y_1y_2,1,y_1y_2,1]]},
\end{aligned}$$

and so on. Rephrased in terms of wall-crossing, this observation implies that the effect of crossing one more wall is just increasing the length of the appropriate continued fraction by two terms. Our preliminary calculation suggests that it is nontrivial to show directly that the ratios of these continued fractions are indeed the L_i's from Gupta's formula.

5.5 The Coefficients Perspective

We started our investigation of $\mathcal{N}_{k,n}$ in a very hands-on manner: by using Sage to expand $\frac{F_{mn+k+1}}{F_{mn+k}}$ as a multivariate power series, observing that they tend to a limit as $m \to \infty$, and finding formulas for the coefficients of the conjectural limit. While one can easily obtain these coefficients by expanding Theorem 5.4.2 using the generalized binomial theorem, we record our result for $\mathcal{N}_{1,2}$ and $\mathcal{N}_{2,2}$ to give an idea of what these coefficients will look like.

Example 5.5.1. Let $c(n,m,\ell)$, $d(n,m,\ell)$, $e(n,m,\ell)$ be the coefficients of $\widehat{y}_1^n\widehat{y}_2^m\widehat{y}_3^\ell$ in $\mathcal{N}_{2,2}$, $\mathcal{N}_{1,2}$ and $\mathcal{T}_2$ respectively. Then for $n \geq m > \ell \geq 0$,

$$c(n,m,\ell) = (-1)^{n+\ell+1}\frac{n}{m(m-1)}\binom{n-1}{\ell}\binom{n-1}{\ell-1}\binom{n-\ell-1}{n-m}.$$

Otherwise, $c(n,m,\ell) = 0$. For $n > m \geq \ell \geq 0$,

$$d(n,m,\ell) = (-1)^{n+\ell+1}\frac{1}{n-1}\binom{n-1}{\ell}\binom{n-1}{\ell-1}\binom{n-\ell-1}{n-m-1},$$

and $d(n,m,\ell) = 0$ otherwise. Lastly, for $n \geq m \geq \ell \geq 1$ such that $n \geq \ell+1$,

$$e(n,m,\ell) = \frac{1}{n-1}\binom{n-1}{\ell}\binom{n-1}{\ell-1}\binom{n-\ell}{m-\ell}.$$

Notice that

$$\frac{c(n,m,\ell)}{d(n,m-1,\ell)} = \frac{n(n-1)}{m(m-1)}.$$

An explicit understanding of these coefficients leads naturally to identities where on the left hand side, we have a sum in Gupta's formula style over some subset of $\mathbb{Z}_{\geq 0}^{\infty}$, and on the right hand side, a nice closed form expression, such as the Narayana numbers in the Kronecker case, or slight modifications of the Narayana numbers when $n > 1$. These identities can be viewed independently of the cluster algebra interpretation, and it would be interesting to gain more insight to these identities. For example, in the Kronecker case, we have the following identity by matching coefficients.

Proposition 5.5.2. Let

$$M(m_1, m_2, \dots) = \sum_{j=1}^{\infty} j m_j, \quad N(m_1, m_2, \dots) = \sum_{j=1}^{\infty} (j-1) m_j.$$

Then

$$\sum_{\substack{(m_1, m_2, \dots) \in \mathbb{Z}_{\geq 0}^{\infty} \\ M(m_1, m_2, \dots) = i \\ N(m_1, m_2, \dots) = j}} \prod_{j=1}^{\infty} \binom{1 - \sum_{k=j+1}^{\infty} 2m_k(k-j)}{m_j} = (-1)^{i+j} \operatorname{Nar}(i-1, j)?$$

Example 5.5.3. For instance, there are two sequences with $M = 6, N = 4$, which contribute to $-\operatorname{Nar}(5,4) = -\frac{1}{5}\binom{5}{4}\binom{5}{3} = -10$. They are $(0,1,0,1,0,0,\dots)$ and $(1,0,0,0,1,0,\dots)$, and their contributions are -3 and -7 respectively.

5.6 Other Questions

Recall from Theorem 2.5.2 that walls of the scattering diagram for acyclic quivers are normal to a root, which warrants the following definition for affine acyclic quivers. Given the scattering diagram associated to an acyclic quiver, we say that a wall is *real* if it is normal to a real root, and *imaginary* if it is normal to an imaginary root. Note that there is only one primitive imaginary root in affine root systems. Theorem 2.5.3 tells us the decorating term on real walls, which leaves open $f_{\mathfrak{d}}$ for the imaginary wall $\mathfrak{d}$.

Question 5.6.1. Let $\mathfrak{d}$ be an imaginary wall in the scattering diagram of the cluster algebra $\widetilde{A}_{n,1}$. What is $f_{\mathfrak{d}}$, the function attached to the wall $\mathfrak{d}$?

In Reading (2020b), Nathan Reading computed the wall function for the unique wall with slope -1 in the scattering diagram for the Kronecker quiver, which is normal to the primitive imaginary root $\delta = (1,1)$. He also computed

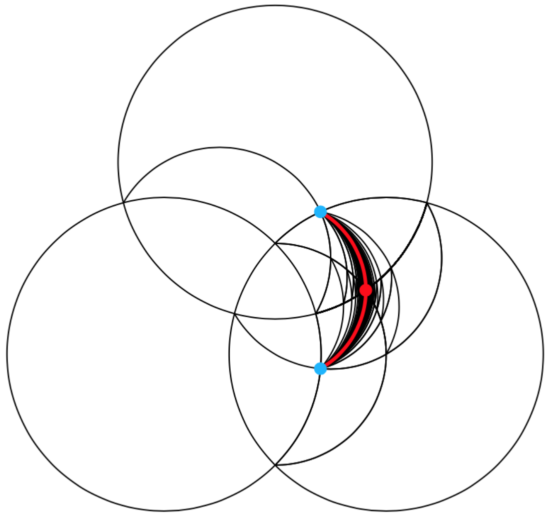


Figure 5.12 Stereographic projection of the scattering diagram of $\widetilde{A}_{2,1}$, figure on page 12 of Reading (2020a)

the function on the imaginary wall of slope -2 in the scattering diagram of $\mathcal{A}(1,4)$, the other affine rank-two cluster algebra. This turned out to be a nontrivial calculation that relies on an understanding of the limit of certain ratios of F-polynomials, which is similar in flavor to our investigation of $\mathcal{N}_{k,n}$. For example, in the Kronecker case, it was enough to understand $\sqrt{\lim_{i\to-\infty} F_i^{2i+3} F_{i+1}^{-2i-1}}$. Since Gupta's formula has a product form, it should be good at dealing with ratios of F-polynomials, which lends the possibility that Gupta's formula might be adept for streamlining this calculation.

In the following example, we calculate the wall function on the wall whose bounding rays are $\mathbf{g}_w$ and the ray $\mathbb{R}_{\geq 0}\begin{bmatrix}-1\\0\\1\end{bmatrix}$, i.e. the top red wall in Figure 5.13 Our calculation relies on the following visualization of the scattering diagram for $\widetilde{A}_{2,1}$ made by Nathan Reading, which is a stereographic projection. In this figure, rays correspond to intersections of walls, which we label with the corresponding cluster variable whose $\mathbf{g}$-vector is the direction of the ray.

Example 5.6.2. We start by noting that

$$f_{\mathfrak{d}}(\widehat{y}_1\widehat{y}_2\widehat{y}_3)^{-2} = \frac{p_\infty(x_1x_3 f_{\mathfrak{d}}(\widehat{y}_1\widehat{y}_2\widehat{y}_3)^{-2})}{x_1x_3}\bigg|_{\text{diag}} = \frac{p_{-\infty}(x_1x_3)}{x_1x_3}\bigg|_{\text{diag}},$$

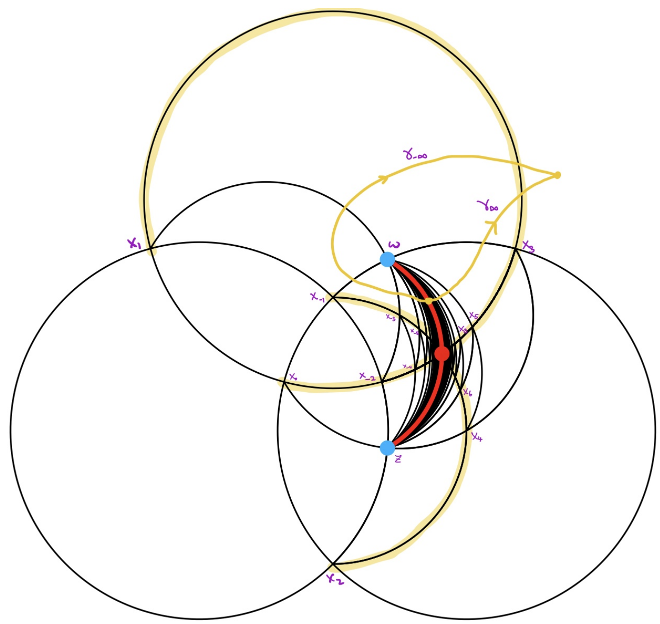


Figure 5.13 Figure 5.12 annotated with the cluster variables whose $\mathbf{g}$-vectors correspond to the rays. On the four yellow highlights emanating from the red dot lie the cluster variables x_m where m is negative odd, negative even, positive odd and nonnegative even

where by diagonal terms we mean monomials $\widehat{y}^{i\delta} = \widehat{y}_1^i \widehat{y}_2^i \widehat{y}_3^i$.

The second equality follows from consistency. Since wall-crossing automorphisms replace monomials by monomials times a power of the wall function and since all walls crossed by γ_∞ and $\gamma_{-\infty}$ are normal to real roots, restricting to diagonal terms and dividing by $x_1 x_3$ recovers $f_{\mathfrak{d}}(\widehat{y}_1 \widehat{y}_2 \widehat{y}_3)^{-2}$ exactly.

We now proceed to compute $\left.\frac{p_{-\infty}(x_1 x_3)}{x_1 x_3}\right|_{\text{diag}}$. Computation shows that for $m \geq 0$,

$$\mathbf{g}_w = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{g}_{-2m-1} = \begin{bmatrix} -m \\ -1 \\ m \end{bmatrix}.$$

So for $m \geq 1$,

$$\mathbf{g}_{-2m-1} = \begin{bmatrix} -m \\ 0 \\ m-1 \end{bmatrix} + \mathbf{g}_w.$$

If we let $Q_m = 2m - 1$, then

$$\mathbf{g}_{-2m-1} Q_{m-1} - \mathbf{g}_{-2m+1} Q_m = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2\mathbf{g}_w.$$

(Label γ_i on the figure.) By Theorem 2.5.4, for $i \geq 0$,

$$p_{\gamma_{-i}}(\mathbf{x}^{\mathbf{g}_w}) = \mathbf{x}^{\mathbf{g}_w} F_w(\widehat{\mathbf{y}}) = \mathbf{x}^{\mathbf{g}_w}(1 + \widehat{y}_2),$$

and

$$p_{\gamma_{-i}}(\mathbf{x}^{\mathbf{g}_{-2i-1}}) = \mathbf{x}^{\mathbf{g}_{-2i-1}} F_{-2i-1}(\widehat{\mathbf{y}}).$$

So

$$p_{\gamma_{-i}}(x_1 x_3) = p_{\gamma_{-i}}(\mathbf{x}^{\mathbf{g}_{-2i-1} Q_{i-1} - \mathbf{g}_{-2i+1} Q_i + 2\mathbf{g}_w}) = x_1 x_3 F_w(\widehat{\mathbf{y}})^2 \frac{F_{-2i-1}(\widehat{\mathbf{y}})^{Q_{i-1}}}{F_{-2i+1}(\widehat{\mathbf{y}})^{Q_i}} = x_1 x_3 (1+\widehat{y}_2)^2 \frac{F_{-2i-1}(\widehat{\mathbf{y}})^{2i-3}}{F_{-2i+1}(\widehat{\mathbf{y}})^{2i-1}}.$$

Therefore,

$$\frac{p_{-\infty}(x_1 x_3)}{x_1 x_3} = (1 + \widehat{y}_2)^2 \lim_{i \to \infty} \frac{F_{-2i-1}(\widehat{\mathbf{y}})^{2i-3}}{F_{-2i+1}(\widehat{\mathbf{y}})^{2i-1}}.$$

To find a formula for F_{-2i-1}, consider the change of initial seed from $\{x_1, x_2, x_3\}$ to $\{x_1, w, x_3\}$. The key observation is that if we require that w always stays in the cluster, then the dynamics of the other two cluster variables is exactly that of the Kronecker quiver. Write $\widehat{y}_1', \widehat{y}_2', \widehat{y}_3'$ for the hatted y's at this cluster seed and write $\tilde{g}_{-m}$ and $\widetilde{F}_{-m}$ for the **g**-vector and F-polynomial of $(-m+1)$-th cluster variable as we apply the mutation sequence $\mu_3\mu_1\mu_3\mu_1\ldots$ to the cluster $\{x_1, w, x_3\}$. Then

$$\tilde{\mathbf{g}}_{-m} = \begin{bmatrix} -m \\ 1 \\ m-1 \end{bmatrix}, \ \widetilde{F}_{1-m}(\widehat{y}_1', \widehat{y}_2', \widehat{y}_3') = \sum_{0\le N\le M\le \ell} \binom{\ell-N}{\ell-M}\binom{M-1}{N} \widehat{y}_1'^{\,\ell-1-N}\widehat{y}_3'^{\,\ell-M}.$$

Some algebra tells us that

$$\widehat{y}_1' = \widehat{y}_1\widehat{y}_3 + \widehat{y}_1, \quad \widehat{y}_3' = \frac{\widehat{y}_2\widehat{y}_3}{1+\widehat{y}_3},$$

from which we can derive

$$F_{-2m-1}(\widehat{y}_1, \widehat{y}_2, \widehat{y}_3) = \sum_{0\le N\le M\le \ell} \binom{\ell-N}{\ell-M}\binom{M-1}{N} \widehat{y}_1^{\,\ell-N}(\widehat{y}_2\widehat{y}_3)^{\ell+1-M}(1+\widehat{y}_2)^{M-N}.$$

As a result,

$$\begin{aligned}
&(1+\widehat{y}_2)^2 \lim_{i\to\infty} \frac{F_{-2i-1}(\widehat{\mathbf{y}})^{2i-3}}{F_{-2i+1}(\widehat{\mathbf{y}})^{2i-1}}\Bigg|_{\mathrm{diag}} \\
&= \lim_{i\to\infty} \frac{\left[\sum_{0\le N\le M\le i+1}\binom{i-N}{\ell-M}\binom{M-1}{N}\widehat{y}_1^{\,i-N}(\widehat{y}_2\widehat{y}_3)^{i+1-M}\right]^{2i-3}}{\left[\sum_{0\le N\le M\le i}\binom{i-N}{\ell-M}\binom{M-1}{N}\widehat{y}_1^{\,i-1-N}(\widehat{y}_2\widehat{y}_3)^{i-M}\right]^{2i-1}}\Bigg|_{\mathrm{diag}} \\
&= \lim_{i\to\infty} \frac{\widetilde{F}_{-i}(\widehat{y}_1, \widehat{y}_2\widehat{y}_3)^{2i-3}}{\widetilde{F}_{-i+1}(\widehat{y}_1, \widehat{y}_2\widehat{y}_3)^{2i-1}}\Bigg|_{\mathrm{diag}} \\
&= (1-\widehat{y}_1\widehat{y}_2\widehat{y}_3)^4,
\end{aligned}$$

where the final equality is due to Theorem 3.4 and Proposition 3.5 of Reading (2020b). This shows that the wall function is $f_{\mathfrak{d}} = (1-\widehat{y}_1\widehat{y}_2\widehat{y}_3)^{-2}$.

Appendix A

The Sequence c_m

We have noted previously that for $r \geq 2$, the sequence of c-vector entries of $\mathcal{A}(r, r)$ is given by the recurrence relation $c_1 = 0$, $c_2 = 1$, $c_{n+2} = rc_{n+1} - c_n$. This is also the Lucas sequence $U_n(r, 1)$. In this appendix, we develop some properties of this sequence which we used without proof in the previous sections.

By solving this recurrence relation, we find that

$$c_m = \frac{1}{\sqrt{r^2 - 4}} \left(r_1^{m-1} + r_2^{m-1} \right), \tag{A.1}$$

where

$$r_1 = \frac{r + \sqrt{r^2 - 4}}{2}, \quad r_2 = \frac{r - \sqrt{r^2 - 4}}{2}.$$

Notice that $r_1 r_2 = 1$.

Proposition A.0.3. For $m \geq 2$,

$$c_m^2 = c_{m-1}c_{m+1} + 1.$$

Proof. This is true for $m = 2$ because $c_1 = 0, c_2 = 1, c_3 = r$. Suppose true for $m - 1$. Then

$$\begin{aligned}
c_{m-1}c_{m+1} + 1 &= c_{m-1}(rc_m - c_{m-1}) + 1 \\
&= rc_m c_{m-1} - (c_{m-1}^2 - 1) \\
&= rc_m c_{m-1} - c_m c_{m-2} \\
&= c_m(rc_{m-1} - c_{m-2}) \\
&= c_m^2.
\end{aligned}$$

□

Proposition A.0.4. For $m \geq 2$,

$$\gcd(c_m, c_{m+1}) = 1.$$

Proof. We proceed by induction. This is true for $m = 2$ since $c_2 = 1$ and $c_3 = r$. Now suppose it is true for $m-1$. Then for $x \neq 1$, $x \mid c_m$ would imply that $x \nmid c_{m-1}$, which means that x cannot be a factor of $c_{m+1} = rc_m - c_{m-1}$. □

Proposition A.0.5. The sequence $\frac{c_m}{c_{m+1}}$ is increasing, and

$$\lim_{m\to\infty} \frac{c_m}{c_{m+1}} = r_2 = \frac{r - \sqrt{r^2-4}}{2}.$$

Proof. By Proposition A.0.3,

$$\frac{c_m}{c_{m+1}} - \frac{c_{m-1}}{c_m} = \frac{c_m^2 - c_{m+1}c_{m-1}}{c_m c_{m+1}} = \frac{1}{c_m c_{m+1}} > 0,$$

which shows that the sequence $\frac{c_m}{c_{m+1}}$ is increasing. We can compute the limit using the closed form expression Equation A.1:

$$\lim_{m\to\infty} \frac{c_m}{c_{m+1}} = \lim_{m\to\infty} \frac{r_1^{m-1} + r_2^{m-1}}{r_1^m + r_2^m} = \lim_{m\to\infty} \frac{r_1^{m-1}}{r_1^m} = \frac{1}{r_1} = r_2.$$

Terms of the form r_2^m vanish in the limit because $0 < r_2 < 1$. □

Proposition A.0.6. Let $3 \leq m \leq n-2$ and $1 \leq w \leq r-1$. Then for any $n \geq 1$,

$$\frac{c_m - wc_{m-1}}{c_{m+1} - wc_m} > \frac{c_n}{c_{n+1}}.$$

Proof. By Proposition A.0.5, it suffices to show that

$$\frac{c_m - wc_{m-1}}{c_{m+1} - wc_m} > r_2,$$

or $r_2(c_{m+1} - wc_m) < c_m - wc_{m-1}$. We obtain this inequality using the closed form expression Equation A.1:

$$r_2(c_{m+1} - wc_m) = r_2(r_1^m - r_2^m - w(r_1^{m-1} - r_2^{m-1})) = r_1^{m-1} - wr_1^{m-2} + wr_2^m - r_2^{m+1},$$

but $wr_2^m - r_2^{m+1} = r_2(wr_2^{m-1} - r_2^m) < wr_2^{m-1} - r_2^m$, so

$$r_2(c_{m+1} - wc_m) < r_1^{m-1} - wr_1^{m-2} + wr_2^{m-1} - r_2^m = c_m - wc_{m-1}$$

as desired. □

Proposition A.0.7.

$$-c_j c_{k-1} + c_{j-1} c_k = c_{k-j}.$$

Proof. We proceed by induction on $k - j$ with k fixed. This is true when $k - j = 0$. By Proposition A.0.3, this is true when $k - j = 1$. By the induction hypothesis, $-c_{j+2}c_{k-1} + c_{j+1}c_k = c_{k-j-2}$ and $-c_{j+1}c_{k-1} + c_j c_k = c_{k-j-1}$. Thus, since $c_j = rc_{j+1} - c_{j+2} = rc_{j-1} - c_{j-2}$, we have $-c_j c_{k-1} + c_{j-1}c_k = c_{k-j}$. □

Appendix B

Proof that Theorems 5.4.2, 5.4.5 and 5.4.7 are Equivalent

Proof of Theorem 5.4.2 ⇔ Theorem 5.4.5. Throughout this proof, for convenience, we will use $\mathcal{N}_{k,n}$ to denote the right hand side of Equation 5.10, rather than the F-polynomial limits.

Let

$$\mathcal{L}_{k,n} = \begin{cases} \frac{1}{\mathcal{N}_{1,n}-(1+y_1)} & \text{if } k = 1, \\ \frac{1}{\frac{1}{\mathcal{N}_{k,n}-1}-\gamma_{k,n}} & \text{if } 1 < k \leq n. \end{cases}$$

It suffices to show that Equation 5.10 is equivalent to

$$\mathcal{L}_{k,n} = \begin{cases} [\overline{\alpha_{1,n}, \beta_{1,n}}] & \text{if } k = 1, \\ [\overline{\beta_{k,n}, \alpha_{k,n}}] & \text{if } 1 < k \leq n. \end{cases}$$

Let us first consider $\mathcal{L}_{1,n}$. Recall that

$$A_{1,n} = P_+(y_2, y_3, \ldots, y_n),\ B_{1,n} = y_2 P_+(y_3, \ldots, y_n).$$

Therefore,

$$\begin{aligned} \frac{1}{\mathcal{L}_{1,n}} &= \mathcal{N}_{1,n} - (1 + y_1) \\ &= \frac{P_+(y_1, \ldots, y_{n+1}) + 2y_2 P_+(y_3, \ldots, y_n) + \sqrt{\Delta_n} - 2(1 + y_1)P_+(y_2, y_3, \ldots, y_n)}{2P_+(y_2, y_3, \ldots, y_n)} \\ &= \frac{-P_-(y_1, \ldots, y_{n+1}) + \sqrt{\Delta_n}}{2P_+(y_2, y_3, \ldots, y_n)}, \end{aligned} \tag{B.1}$$

so

$$\begin{aligned}\mathcal{L}_{1,n} &= \frac{2P_+(y_2, y_3, \ldots, y_n)(\sqrt{\Delta_n} + P_-(y_1, \ldots, y_{n+1}))}{\Delta_n - P_-(y_1, \ldots, y_{n+1})^2} \\ &= \frac{2P_+(y_2, y_3, \ldots, y_n)(\sqrt{\Delta_n} + P_-(y_1, \ldots, y_{n+1}))}{4y_1^2 y_2 \cdots y_{n+1} P_+(y_2, y_3, \ldots, y_n)} \\ &= \alpha_{1,n} + \frac{\sqrt{\Delta_n} - P_-(y_1, \ldots, y_{n+1})}{2y_1^2 y_2 \cdots y_{n+1}}.\end{aligned}$$

Continuing, we compute that

$$\begin{aligned}\frac{2y_1^2 y_2 \cdots y_{n+1}}{\sqrt{\Delta_n} - P_-(y_1, \ldots, y_{n+1})} &= \frac{2y_1^2 y_2 \cdots y_{n+1}(\sqrt{\Delta_n} + P_-(y_1, \ldots, y_{n+1})}{4y_1^2 y_2 \cdots y_{n+1} P_+(y_2, y_3, \ldots, y_n)} \\ &= \frac{\sqrt{\Delta_n} + P_-(y_1, \ldots, y_{n+1})}{2P_+(y_2, y_3, \ldots, y_n)} \\ &= \beta_{1,n} + \frac{1}{\mathcal{L}_{1,n}}\end{aligned}$$

Combining our computation above, we get

$$\mathcal{L}_{1,n} = \alpha_{1,n} + \cfrac{1}{\beta_{1,n} + \cfrac{1}{\mathcal{L}_{1,n}}},$$

which proves that

$$\mathcal{L}_{1,n} = [\overline{\alpha_{1,n}, \beta_{1,n}}].$$

We now move on to perform a similar computation for $\mathcal{L}_{k,n}$. Observe that

$$\begin{aligned}\mathcal{N}_{k,n} - 1 &= \frac{P_+(y_1, \ldots, y_{n+1}) + 2(B_{k,n} - A_{k,n}) + \sqrt{\Delta_n}}{2A_{k,n}} \\ &= \frac{P_+(y_1, \ldots, y_{n+1}) - 2P_+(y_1, \ldots, y_{k-1}) + \sqrt{\Delta_n}}{2A_{k,n}}.\end{aligned}$$

So

$$\begin{aligned}\frac{1}{\mathcal{N}_{k,n} - 1} &= \frac{2A_{k,n}}{P_+(y_1, \ldots, y_{n+1}) - 2P_+(y_1, \ldots, y_{k-1}) + \sqrt{\Delta_n}} \\ &= \frac{2A_{k,n}(-P_+(y_1, \ldots, y_{n+1}) + 2P_+(y_1, \ldots, y_{k-1}) + \sqrt{\Delta_n})}{\Delta_n - (P_+(y_1, \ldots, y_{n+1}) - 2P_+(y_1, \ldots, y_{k-1}))^2}. \qquad \text{(B.2)}\end{aligned}$$

We may calculate that

$$\begin{aligned}
&\Delta_n - (P_+(y_1,\ldots,y_{n+1}) - 2P_+(y_1,\ldots,y_{k-1}))^2 \\
&= (P_+(y_1,\ldots,y_{k-1}) + y_1y_2\cdots y_kP_-(y_{k+1},\ldots,y_{n+1}))^2 + 4y_1^2y_2\cdots y_{n+1}P_+(y_2,\ldots,y_n) \\
&\quad - (-P_+(y_1,\ldots,y_{k-1}) + y_1y_2\cdots y_kP_+(y_{k+1},\ldots,y_{n+1}))^2 \\
&= \big(y_1y_2\cdots y_k\cdot 2P_+(y_{k+1},\ldots,y_n)\big)\big(2P_+(y_1,\ldots,y_{k-1}) - 2y_1y_2\cdots y_{n+1}\big) \\
&\quad + 4y_1^2y_2\cdots y_{n+1}P_+(y_2,\ldots,y_n) \\
&= 4y_1y_2\cdots y_k\big(P_-(y_1,\ldots,y_{k-1})P_+(y_{k+1},\ldots,y_{n+1}) - y_{k+1}y_{k+2}\cdots y_{n+1}\big) \\
&= 4y_1y_2\cdots y_kA_{k,n}. \qquad \text{(B.3)}
\end{aligned}$$

In particular, when specializing to $k = 1$, this calculation suggests

$$\Delta_n - (P_+(y_1,\ldots,y_{n+1}) - 2)^2 = 4y_1P_+(y_2,\ldots,y_n). \qquad \text{(B.4)}$$

The identity B.3 allows us to simplify the denominator of Equation B.2 get

$$\begin{aligned}
\frac{1}{\mathcal{N}_{k,n} - 1} &= \frac{-P_+(y_1,\ldots,y_{n+1}) + 2P_+(y_1,\ldots,y_{k-1}) + \sqrt{\Delta_n}}{2y_1y_2\cdots y_k} \qquad \text{(B.5)} \\
&= \gamma_{k,n} + \frac{-P_+(y_1,\ldots,y_{n+1}) + 2 + \sqrt{\Delta_n}}{2y_1y_2\cdots y_k},
\end{aligned}$$

and so

$$\begin{aligned}
\mathcal{L}_{k,n} &= \frac{1}{\frac{1}{\mathcal{N}_{k,n}-1} - \gamma_{k,n}} \\
&= \frac{2y_1y_2\cdots y_k}{-P_+(y_1,\ldots,y_{n+1}) + 2 + \sqrt{\Delta_n}} \\
&= \frac{2y_1y_2\cdots y_k(P_+(y_1,\ldots,y_{n+1}) - 2 + \sqrt{\Delta_n})}{\Delta_n - (P_+(y_1,\ldots,y_{n+1}) - 2)^2} \\
&= \frac{2y_1y_2\cdots y_k(P_+(y_1,\ldots,y_{n+1}) - 2 + \sqrt{\Delta_n})}{4y_1P_+(y_2,\ldots,y_n)} \quad \text{by Equation B.4} \\
&= \beta_{k,n} + \frac{y_2\cdots y_k(-P_+(y_1,\ldots,y_{n+1}) + 2 + \sqrt{\Delta_n})}{2P_+(y_2,\ldots,y_n)}.
\end{aligned}$$

Continuing, we compute that

$$\begin{aligned}
&\frac{2P_+(y_2,\ldots,y_n)}{y_2\cdots y_k(-P_+(y_1,\ldots,y_{n+1})+2+\sqrt{\Delta_n})}\\
&= \frac{2P_+(y_2,\ldots,y_n)(\sqrt{\Delta_n}+P_+(y_1,\ldots,y_{n+1})-2)}{y_2\cdots y_k(\Delta_n-(P_+(y_1,\ldots,y_{n+1})-2)^2)}\\
&= \frac{2P_+(y_2,\ldots,y_n)(\sqrt{\Delta_n}+P_+(y_1,\ldots,y_{n+1})-2)}{4y_1y_2\cdots y_kP_+(y_2,\ldots,y_n)} \qquad \text{by Equation B.4}\\
&= \frac{\sqrt{\Delta_n}+P_+(y_1,\ldots,y_{n+1})-2}{2y_1y_2\cdots y_k}\\
&= \alpha_{k,n}+\frac{\sqrt{\Delta_n}-P_+(y_1,\ldots,y_{n+1})+2}{2y_1y_2\cdots y_k}\\
&= \alpha_{k,n}+\frac{1}{\mathcal{L}_{k,n}}.
\end{aligned}$$

Combining the previous calculations, we have shown that for $1 < k \le n$,

$$\mathcal{L}_{k,n} = \beta_{k,n} + \frac{1}{\alpha_{k,n}+\frac{1}{\mathcal{L}_{k,n}}},$$

which affirms our claim that

$$\mathcal{L}_{k,n} = [\overline{\beta_{k,n},\alpha_{k,n}}].$$

This concludes our proof of the equivalence between Theorem 5.4.2 and Theorem 5.4.5. □

Proof of Theorem 5.4.2 ⇔ Theorem 5.4.7. Again we will use $\mathcal{N}_{k,n}$ to denote the right hand side of Equation 5.10, rather than the F-polynomial limits. We shall also reuse the notation $\mathcal{L}_{k,n}$ from the previous proof to mean morally similar, but slightly different things.

Let

$$\mathcal{L}_{k,n} = \begin{cases} \mathcal{N}_{1,n}-(1+y_1) & \text{if } k=1,\\ \frac{y_2y_3\cdots y_k}{\mathcal{N}_{k,n}-1} - P_+(y_2,\ldots,y_{k-1}) & \text{if } 1<k\le n. \end{cases}$$

In order to show that $\mathcal{N}_{k,n}$ can be expressed in terms of the continued fractions of Theorem 5.4.7, it suffices to show that

$$\mathcal{L}_{1,n} = \frac{y_1y_2\cdots y_{n+1}}{P_+(y_2,\ldots,y_n)+\frac{1}{y_1+\mathcal{L}_{1,n}}}, \tag{B.6}$$

and

$$\mathcal{L}_{k,n} = \cfrac{1}{y_1 + \cfrac{y_1 y_2 \cdots y_{n+1}}{P_+(y_2, \ldots, y_n) + \mathcal{L}_{k,n}}}. \tag{B.7}$$

We first prove Equation B.6. Equation B.1 from the previous proof says that

$$\mathcal{L}_{1,n} = \frac{-P_-(y_1, \ldots, y_{n+1}) + \sqrt{\Delta_n}}{2P_+(y_2, y_3, \ldots, y_n)},$$

So

$$\begin{aligned}
\frac{y_1 y_2 \cdots y_{n+1}}{\mathcal{L}_{1,n}} &= \frac{2y_1 y_2 \cdots y_{n+1} P_+(y_2, y_3, \ldots, y_n)}{-P_-(y_1, \ldots, y_{n+1}) + \sqrt{\Delta_n}} \\
&= \frac{2y_1 y_2 \cdots y_{n+1} P_+(y_2, y_3, \ldots, y_n)(\sqrt{\Delta_n} + P_-(y_1, \ldots, y_{n+1}))}{\Delta_n - P_-(y_1, \ldots, y_{n+1})^2} \\
&= \frac{\sqrt{\Delta_n} + P_-(y_1, \ldots, y_{n+1})}{2y_1} \\
&= P_+(y_2, \ldots, y_n) + \frac{\sqrt{\Delta_n} + P_-(y_1, \ldots, y_{n+1}) - 2y_1 P_+(y_2, \ldots, y_n)}{2y_1} \\
&= P_+(y_2, \ldots, y_n) + \frac{\sqrt{\Delta_n} + 2 - P_+(y_1, \ldots, y_{n+1})}{2y_1},
\end{aligned}$$

where the third equality uses the definition that $\Delta_n = P_-(y_1, \ldots, y_{n+1})^2 + 4y_1^2 y_2 \cdots y_{n+1} P_+(y_2, \ldots, y_n)$. Using Equation B.4,

$$\begin{aligned}
\frac{1}{\frac{y_1 y_2 \cdots y_{n+1}}{\mathcal{L}_{1,n}} - P_+(y_2, \ldots, y_n)} &= \frac{2y_1}{\sqrt{\Delta_n} + P_-(y_1, \ldots, y_{n+1}) - 2y_1 P_+(y_2, \ldots, y_n)} \\
&= \frac{2y_1(\sqrt{\Delta_n} - P_-(y_1, \ldots, y_{n+1}) + 2y_1 P_+(y_2, \ldots, y_n))}{\Delta_n - (P_-(y_1, \ldots, y_{n+1}) - 2y_1 P_+(y_2, \ldots, y_n))^2} \\
&= \frac{\sqrt{\Delta_n} - P_-(y_1, \ldots, y_{n+1}) + 2y_1 P_+(y_2, \ldots, y_n)}{2P_+(y_2, \ldots, y_n)} \\
&= y_1 + \mathcal{L}_{1,n},
\end{aligned}$$

as desired.

We now move on to $\mathcal{L}_{k,n}$ where $1 < k \leq n$. Continuing the calculation of Equation B.5, we get that

$$\begin{aligned}
\mathcal{L}_{k,n} &= \frac{-P_+(y_1,\ldots,y_{n+1}) + 2P_+(y_1,\ldots,y_{k-1}) + \sqrt{\Delta_n}}{2y_1} - P_+(y_2,\ldots,y_{k-1}) \\
&= \frac{2 - P_+(y_1,\ldots,y_{n+1}) + \sqrt{\Delta_n}}{2y_1}.
\end{aligned}$$

So by Equation B.4,

$$\begin{aligned}
\frac{1}{\mathcal{L}_{k,n}} &= \frac{2y_1}{2 - P_+(y_1,\ldots,y_{n+1}) + \sqrt{\Delta_n}} \\
&= \frac{2y_1(\sqrt{\Delta_n} - 2 + P_+(y_1,\ldots,y_{n+1}))}{\Delta_n - (P_+(y_1,\ldots,y_{n+1}) - 2)^2} \\
&= \frac{\sqrt{\Delta_n} - 2 + P_+(y_1,\ldots,y_{n+1})}{2P_+(y_2,\ldots,y_n)} \\
&= y_1 + \frac{\sqrt{\Delta_n} - P_-(y_1,\ldots,y_{n+1})}{2P_+(y_2,\ldots,y_n)}.
\end{aligned}$$

We can calculate that

$$\begin{aligned}
\frac{2y_1y_2\cdots y_{n+1}P_+(y_2,\ldots,y_n)}{\sqrt{\Delta_n} - P_-(y_1,\ldots,y_{n+1})} &= \frac{2y_1y_2\cdots y_{n+1}P_+(y_2,\ldots,y_n)(\sqrt{\Delta_n} + P_-(y_1,\ldots,y_{n+1}))}{\Delta_n - P_-(y_1,\ldots,y_{n+1})^2} \\
&= \frac{2y_1y_2\cdots y_{n+1}P_+(y_2,\ldots,y_n)(\sqrt{\Delta_n} + P_-(y_1,\ldots,y_{n+1}))}{4y_1^2y_2\cdots y_{n+1}P_+(y_2,\ldots,y_n)} \\
&= \frac{\sqrt{\Delta_n} + P_-(y_1,\ldots,y_{n+1})}{2y_1} \\
&= P_+(y_2,\ldots,y_n) + \frac{\sqrt{\Delta_n} + P_-(y_1,\ldots,y_{n+1}) - 2y_1P_+(y_2,\ldots,y_n)}{2y_1} \\
&= P_+(y_2,\ldots,y_n) + \mathcal{L}_{k,n},
\end{aligned}$$

where the second equality again simply uses the definition of Δ_n. This allows us to conclude that for $1 < k \le n$,

$$\frac{1}{\mathcal{L}_{k,n}} = y_1 + \frac{y_1y_2\cdots y_{n+1}}{P_+(y_2,\ldots,y_n) + \mathcal{L}_{k,n}},$$

as desired. $\square$

Bibliography

Ariki, Susumu, Hiraku Nakajima, Yoshihisa Saito, Ken-ichi Shinoda, Toshiaki Shoji, and Toshiyuki Tanisaki, eds. 2010. *Proceedings of the 10th International Conference on Representation Theory of Algebraic Groups and Quantum Groups*, *Contemporary Mathematics*, vol. 565. AMS.

Berstel, Jean, Aaron Lauve, Christophe Reutenauer, and Franco Saliola. 2008. *Combinatorics on Words: Christoffel Words and Repetitions in Words*, *CRM Monograph Series*, vol. 27. American Mathematical Society and Centre de Recherches Mathématiques.

Caldero, Philippe, and Andrei Zelevinsky. 2006. Laurent expansions in cluster algebras via quiver representations. *Moscow Mathematical Journal* (3). doi:10.17323/1609-4514-2006-6-3-411-429.

Canakci, Ilke, and Ralf Schiffler. 2017. Cluster algebras and continued fractions. *Compositio Mathematica* 154(3):565–593. doi:10.1112/s0010437x17007631.

Cerulli Irelli, Giovanni. 2011. Cluster algebras of type $A_2^{(1)}$. *Algebras and Representation Theory* 15(5):977–1021. doi:10.1007/s10468-011-9275-5.

Cheung, Man Wai, Mark Gross, Greg Muller, Gregg Musiker, Dylan Rupel, Salvatore Stella, and Harold Williams. 2017. The greedy basis equals the theta basis: a rank two haiku. *Journal of Combinatorial Theory, Series A* 145:150–171. doi:10.1016/j.jcta.2016.08.004.

Fomin, Sergey, Michael Shapiro, and Dylan Thurston. 2008. Cluster algebras and triangulated surfaces. Part I: Cluster complexes. *Acta Mathematica* 201(1):83–146. doi:10.1007/s11511-008-0030-7.

Fomin, Sergey, and Dylan Thurston. 2018. Cluster algebras and triangulated surfaces part ii: Lambda lengths. *Memoirs of the American Mathematical Society* 255(1223):2891–2944. doi:10.1090/memo/1223.

Fomin, Sergey, and Andrei Zelevinsky. 2002. Cluster algebras I: foundations. *Journal of the American Mathematical Society* 15(2):497–529.

———. 2003a. Cluster algebras II: finite type classification. *Inventiones Mathematicae* 154(1):63–121. doi:10.1007/s00222-003-0302-y.

———. 2003b. Y-systems and generalized associahedra. *Annals of Mathematics* 158(3):977–1018.

———. 2007. Cluster algebras IV: coefficients. *Compositio Mathematica* 143(1):112–164.

Gross, Mark, Paul Hacking, Sean Keel, and Maxim Kontsevich. 2018. Canonical bases for cluster algebras. *Journal of the American Mathematical Society* 31(2):497–608.

Gupta, Meghal. 2018. A formula for F-Polynomials in terms of $\mathbf{c}$-Vectors and Stabilization of F-Polynomials. arXiv:1812.01910.

Hardy, G. H., and Edward M. Wright. 2008. *An Introduction to the Theory of Numbers*. Oxford mathematics, OUP Oxford.

Kac, Victor G. 1990. *Infinite-Dimensional Lie Algebras*. Cambridge University Press, 3rd ed. doi:10.1017/CBO9780511626234.

Keller, Bernhard. 2012. Cluster algebras and derived categories. *Proceedings of the GCOE conference, Derived Categories 2011 Tokyo* arXiv:1202.4161.

Lee, Kyungyong. 2012. On cluster variables of rank two acyclic cluster algebras. *Annals of Combinatorics* 16(2):305–317. doi:10.1007/s00026-012-0134-9.

Lee, Kyungyong, Li Li, and Ba Nguyen. 2017. New Combinatorial Formulas for Cluster Monomials of Type A Quivers. *The Electronic Journal of Combinatorics* 24(2):P2.42. doi:10.37236/6464.

Lee, Kyungyong, Li Li, and Andrei Zelevinsky. 2014. Greedy elements in rank 2 cluster algebras. *Selecta Mathematica* 20(1):57–82.

Lee, Kyungyong, and Ralf Schiffler. 2013. A combinatorial formula for rank 2 cluster variables. *Journal of Algebraic Combinatorics* 37(1):67–85.

———. 2015. Positivity for cluster algebras. *Annals of Mathematics* 182(1):73–125.

Lin, Feiyang. 2020. F-polynomials for the r-Kronecker quiver. URL http://www-users.math.umn.edu/~reiner/REU/REU2020notes/Problem3_REUreport.pdf.

Morier-Genoud, Sophie, and Valentin Ovsienko. 2020. q-deformed rationals and q-continued fractions. *Forum of Mathematics, Sigma* 8:e13. doi:10.1017/fms.2020.9.

Musiker, Gregg. 2019. Applications of new F-polynomial formulas in terms of $\mathbf{c}$-vectors. Personal Communication.

———. 2020. REU Problem 3: Questions on generating functions via cluster algebras. URL http://www-users.math.umn.edu/~reiner/REU/REU2020notes/Problem3_Musiker.pdf.

Musiker, Gregg, and James Propp. 2007. Combinatorial interpretations for rank-two cluster algebras of affine type. *The Electronic Journal of Combinatorics* 14(1):R15.

Musiker, Gregg, and Lauren Williams. 2013. Matrix formulae and skein relations for cluster algebras from surfaces. *International Mathematics Research Notices* 2013(13):2891–2944. doi:10.1093/imrn/rns118.

Nagao, Kentaro. 2013. Donaldson–Thomas theory and cluster algebras. *Duke Mathematical Journal* 162(7):1313–1367. doi:10.1215/00127094-2142753.

Nakanishi, Tomoki. 2021. Cluster patterns and scattering diagrams arXiv:2103.16309.

Nakanishi, Tomoki, and Andrei Zelevinsky. 2012. On tropical dualities in cluster algebras. In Ariki et al. (2010), 217–226. doi:10.1090/conm/565/11159.

Reading, Nathan. 2018. To scatter or to cluster. URL https://nreadin.math.ncsu.edu/papers/RomeTalk.pdf.

———. 2020a. Cluster scattering diagrams of affine type. Personal Communication.

———. 2020b. A combinatorial approach to scattering diagrams. *Algebraic Combinatorics* 3(3):603–636. doi:10.5802/alco.107.

Reading, Nathan, and Salvatore Stella. 2020. An affine almost positive roots model. *Journal of Combinatorial Algebra* 4(1):1–59. doi:10.4171/JCA/37.

Sherman, Paul, and Andrei Zelevinsky. 2004. Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. *Moscow mathematical journal* 4(4):947–974.

Williams, Lauren K. 2014. Cluster algebras: an introduction. *Bulletin of the American Mathematical Society* 51:1–26. doi:10.1090/S0273-0979-2013-01417-4.

Zelevinsky, Andrei. 2006. Semicanonical basis generators of the cluster algebra of type $A_1^{(1)}$. arXiv:0606775.